# SOME BOUNDARY VALUE PROBLEMS FOR LINEAR DIFFERENTIAL SYSTEMS[1]

W. J. COLES

1. **Introduction.** Let $A(t)$ and $f(t)$ be $n\times n$ and $n\times 1$ matrices, respectively, continuous on an interval $[a, b]$. In [1], J. B. Garner and L. P. Burton consider the boundary value problems

$$(1) \qquad y' = Ay + f, \qquad y_i(a) = \beta_i \ (1 \leqq i < n), \qquad y_n(b) = \beta_n$$

and

$$(2) \quad y' = Ay + f, \quad y_1(a) = \beta_1, \quad y_i(c) = \beta_i \ (1 < i < n,\ a < c < b), \quad y_n(b) = \beta_n,$$

and prove:

THEOREM A. *If, for each $i$ and $j$ ($1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i \neq j$), $a_{ij}a_{in}a_{jn} > 0$ and $a_{in}a_{ni} > 0$ on $[a, b]$, the problem* (1) *has a unique solution*;

THEOREM B. *Under certain conditions on $A(t)$, too lengthy to give here, the problem* (2) *has a unique solution.*

The authors note that Theorem A has a dual in which the roles of $a$ and $b$ are interchanged, provided that $a_{ij}a_{in}a_{jn} < 0$ is assumed.

The purpose here is to obtain theorems corresponding to Theorem A and its dual, with considerably less restriction on $A(t)$, and to use these results to obtain as a direct consequence a theorem corresponding to Theorem B.

2. **The two-point problem.** As usual, we rephrase the problem in terms of the homogeneous system. Let $N$ be fixed ($1 \leqq N \leqq n$); let $Q = (\delta_{iN}\delta_{jN})$, and let $P = E - Q$ ($E$ being the $n\times n$ identity); let $\beta = \operatorname{col}(\beta_i)$. Let $z(t)$ be a solution of $y' = Ay + f$ which does not satisfy

$$(3) \qquad y' = Ay + f, \qquad Py(a) + Qy(b) = \beta.$$

If $X$ is any nonsingular solution of $X' = AX$, the general solution of $y' = Ay + f$ can be written in the form $Xc + z$, and our boundary condition reduces to

$$[PX(a) + QX(b)]\cdot c = \beta - Pz(a) - Qz(b) \neq 0.$$

Thus (3) has a unique solution if and only if the equation $X' = AX$

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has a nonsingular solution for which $PX(a)+QX(b)$ is nonsingular. We may assume that $X(a)=E$; our condition is then that $x_N(b)\neq 0$ if $x'=Ax$ and $x_i(a)=\delta_{iN}$ $(1\leqq i\leqq n)$.

For convenience we list the following conditions and definitions.

(0) $a_{ii}(t)\equiv 0$ $(1\leqq i\leqq n)$.

(I) For $N$ fixed $(1\leqq N\leqq n)$, there exist $K\neq N$ $(1\leqq K\leqq n)$ and $m_K$ $(1\leqq m_K\leqq n-1)$ such that no product $a_{Kj_1}(t_0)a_{j_1j_2}(t_1)\cdots a_{j_mN}(t_m)$, with at most $m_K+1$ factors, changes sign on $a\leqq t_i\leqq b$ $(0\leqq i\leqq m)$, each such product has the same sign, and one such product, with at most $m_K$ factors, is nonzero at $t=a$. Let $s_{KN}$ be 1 or $-1$, according as this last product is positive or negative at $t=a$; let $s_{NN}=1$.

(II) For $N$ fixed, (I) holds for each $K\neq N$; $s_{KN}a_{NK}(t)\geqq 0$; the $m_K$'s may be taken equal.

(III) $x'=Ax$ and $x_i(a)=\delta_{iN}$ $(1\leqq i\leqq n)$.

LEMMA 1. *If $x'=Ax$ and* (0) *holds, and if, for a fixed $j_i$, $\sigma_i$ is the set of integers including* $1,\cdots,n$, *but excluding $j_i$, then*

$$\text{(4i)}\quad x_{j_0}(t_0) = x_{j_0}(a) + \sum_{j_1\in\sigma_0}\int_a^{t_0} a_{j_0j_1}(t_1)x_{j_1}(t_1)dt_1;$$

$$\begin{aligned}\text{(4ii)}\quad x_{j_0}(t_0) = x_{j_0}(a) &+ \sum_{h=1}^{m}\sum_{j_1\in\sigma_0}\cdots\sum_{j_h\in\sigma_{h-1}}\int_a^{t_0}dt_1\cdots\\ &\cdot\int_a^{t_{h-2}}dt_{h-1}\int_a^{t_{h-1}}x_{j_h}(a)\prod_{i=0}^{h-1}a_{j_ij_{i+1}}(t_{i+1})dt_h\\ &+\sum_{j_1\in\sigma_0}\cdots\sum_{j_{m+1}\in\sigma_m}\int_a^{t_0}dt_1\cdots\\ &\cdot\int_a^{t_{m-1}}dt_m\int_a^{t_m}x_{j_{m+1}}(t_{m+1})\prod_{i=0}^{m}a_{j_ij_{i+1}}(t_{i+1})dt_{m+1},\qquad m\geqq 1.\end{aligned}$$

PROOF. Integration of $x'_{j_0}$ gives (4i) and, in fact, a similar expression for each $x_{j_i}$. Substituting these expressions into the right-hand side of (4i), and continuing the process, gives (4ii).

LEMMA 2. *If* (0), (I), *and* (III) *hold, there is a $\delta>0$ such that $s_{KN}x_K(t)>0$ on $(a, a+\delta)$.*

PROOF. In (4), let $j_0=K$ and $j_{m+1}=N$. By (III), the first term in each of (4i) and (4ii) is zero. Since all possible products with $m+1$ factors and of the type in (I) occur in the last set of terms in (4i) or (4ii), proper choice of $m$ will cause to appear a product involving $x_N$ which is nonzero at $a$. This term, and indeed all terms involving $x_N$, have the sign of $s_{KN}$. All other terms in the last set are zero at $a$; all nonzero terms in the second set of terms in (4ii) involve $x_N(a)$, and so have the sign of $s_{KN}$. Hence, for small positive $\delta$, $s_{KN}x_K(t)>0$ on $(a, a+\delta)$.

COROLLARY 1. *If* (0), (II) *and* (III) *hold, there is a* $\delta>0$ *such that*

$$s_{iN}x_i(t) > 0 \qquad (1 \leqq i \leqq n) \tag{5}$$

*on* $(a, a+\delta)$.

LEMMA 3. *If* (0), (II) *and* (III) *hold, then* (5) *holds on* $(a, b]$.

PROOF. By Corollary 1, there is a $\delta$ such that $0<\delta<b-a$ and for which (5) holds on $(a, a+\delta)$. Let $a<c<a+\delta$, and let $P(t)= x_1(t) \cdots x_n(t)$. Then

$$P(t) = P(c) \exp \int_c^t \sum_{k,j=1;k\neq j}^{n} a_{kj}x_j/x_k dt$$

on $(c, a+\delta)$. Now, for each $k$ and $j$ such that $k\neq j$, we have $a_{kj}x_j/x_k \geqq 0$. Indeed, if $k=N$ or $j=N$ this follows from the condition $s_{iN}a_{Ni}(t) \geqq 0$ in (II) and from the conclusion of Lemma 2. If $j\neq N$ and $k\neq N$, let $m$ be the common value of the $m_i$'s in (II). There is a product $P_{jN}$ of the form in (I) (with $K=j$), with at most $m$ factors, such that $s_{jN}P_{jN}(a)>0$. Since $a_{kj}P_{jN}$ is a product like those in (I), with at most $m+1$ factors, we have $s_{kN}a_{kj}P_{jN}\geqq 0$. Hence we can write $a_{kj}x_j/x_k = [a_{kj}P_{jN}(a)/x_k]\cdot[x_j/P_{jN}(a)]$, with each factor non-negative. Thus the statement is verified. From this, $|P(t)|\geqq|P(c)|$ on $(c, a+\delta)$. Since the inequality must hold even for $t=a+\delta$, then $P(a+\delta)\neq 0$.

Now let $\Delta$ be the lub of the set of $\delta$'s such that $0<\delta<b-a$ and for which (5) holds on $(a, a+\delta)$. Clearly (5) holds on $(a, a+\Delta)$, and so (by the above argument) (5) holds at $t=a+\Delta$. Unless $\Delta=b-a$, the continuity of $P(t)$ gives a contradiction to the lub property of $\Delta$. This completes the proof.

The following theorem, corresponding to Theorem A, is now almost immediate.

THEOREM 1. *If* (II) *holds for products excluding the* $a_{ii}(t)$*'s, the problem* (3) *has a unique solution.*

PROOF. The proof depends only on Lemma 3, and so we must show that Lemma 3 holds even without condition (0). To this end, let $z(t)=G(t)x(t)$, where $x(t)$ satisfies (III) and $G(t)$ is the diagonal matrix for which $g_{ii}(t)=\exp\int_a^t -a_{ii}(s)ds$. Then $z'=B(t)z$, where $b_{ii}(t)\equiv 0$ and, for $i\neq j$, $b_{ij}(t)=a_{ij}(t)\exp\int_a^t[a_{jj}(s)-a_{ii}(s)]ds$. Thus if (II) holds for

$A(t)$ then (II) holds for $B(t)$, with the same $s_{iN}$'s; if (III) holds for $x(t)$ then (III) holds for $z(t)$; and (0) holds for $B(t)$. Thus Lemma 3 as it stands applies to $z(t)$, and also, since $x_i(t)=z_i(t)\exp\int_a^t a_{ii}(s)ds$, to $x(t)$. Thus condition (0) can be eliminated from Lemma 3, and the proof is complete.

The theorem corresponding to the dual of Theorem A is contained in the following statements. Corresponding to (I), (II), and (III) we have:

(I′) As (I), except that the products with an odd number of factors and those with an even number of factors differ in sign, and there is at least one product, say with $r$ factors ($r\leqq m_K$), which is nonzero at $b$. Let $(-1)^r s'_{KN}$ be 1 or $-1$ according as this last product is positive or negative at $b$. Let $s'_{NN}=1$.

(II′) For some fixed $N$, (I) holds for each $K\neq N$ $(1\leqq K\leqq n)$; $a_{NK}s'_{KN}\leqq 0$; and the $m_K$'s may all be taken equal.

(III′) $x'=Ax$, and $x_i(b)=\delta_{iN}$ $(1\leqq i\leqq n)$.

LEMMA 2′. *If* (0), (I′) *and* (III′) *hold, there is a* $\delta>0$ *such that* $s'_{KN}x_K(t)>0$ *on* $(b-\delta, b)$.

PROOF. The proof is like that of Lemma 2, using (4). Alternatively, let $A(t)=A(b)$ for $t>b$; let $s=2b-t$, $B(s)=-A(t)$, and $w(s)=x(t)$; and apply Lemma 2 directly to the system $w'(s)=B(s)w(s)$ on the interval $[b, 2b-a]$.

LEMMA 3′. *If* (0), (II′) *and* (III′) *hold, then*

$$s'_{iN}x_i(t) > 0 \qquad (1 \leqq i \leqq n) \tag{6}$$

*holds on* $[a, b)$.

THEOREM 1′. *If* (II′) *holds for products excluding the* $a_{ii}(t)$*'s, the problem*

$$y' = Ay + f, \qquad Qy(a) + Py(b) = \beta \tag{3′}$$

*has a unique solution.*

Theorem 1 implies a stronger version of Theorem A, since the hypotheses of Theorem A imply (II) for each $N$. Further, coefficient matrices with vanishing entries can be treated; in particular, known theorems (e.g., see [2] and [3]) for the $n$th order scalar case are implied.

## 3. The three-point problem.

Let $a<c<b$.

THEOREM 2. *Let* $M$ *and* $N$ *be fixed* $(1\leqq M\leqq n,\ 1\leqq N\leqq n,\ M\neq N)$.

*Let* (II) *hold on* $[c, b]$ *and* (II′) *hold on* $[a, c]$ *for products excluding the* $a_{ii}(t)$*'s, for* $N$ *and also for* $M$, *with* $s_{MN}s'_{NM}=-1$. *Let* $R=(\delta_{iM}\delta_{jM})$, $Q=(\delta_{iN}\delta_{jN})$, *and* $S=E-R-Q$. *Then the problem*

$$y' = Ay + f, \qquad Qy(a) + Sy(c) + Ry(b) = \beta \tag{7}$$

*has a unique solution.*

Proof. Let $X'=AX$, $X(c)=E$; it suffices to show that $QX(a)+SX(c)+RX(b)$ is nonsingular, the determinant in question being $x_{NN}(a)x_{MM}(b)-x_{NM}(a)x_{MN}(b)$. By Lemma 3, $x_{MM}(b)$ and $s_{MN}x_{MN}(b)$ are positive; by Lemma 3′, $x_{NN}(a)$ and $s'_{NM}x_{NM}(a)$ are positive; hence the determinant is positive.

It is a matter of detail to verify that the hypotheses of Theorem B imply those of Theorem 2. As in the two-point case, coefficient matrices with vanishing entries, and in particular the scalar case (e.g., Theorem 2 in [2]), are allowed.

## References

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University of Utah