A SIMPLE SET WHICH IS NOT EFFECTIVELY SIMPLE

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For each e, let f_e be the partial recursive function

$$U(\mu y T_1(e, n, y)),$$

and let W_e be the range of f_e. Then $W_0, W_1, W_2, \cdots$ is the Kleene enumeration of the recursively enumerable sets. Post [5] calls a recursively enumerable set simple if its complement is infinite but does not contain any infinite, recursively enumerable set. Raymond Smullyan calls a recursively enumerable set W effectively simple if its complement is infinite, and if there is a partial recursive function f such that for each e, if W_e is contained in the complement of W, then $f(e)$ is defined and is greater than the cardinality of W_e.[2] Clearly, an effectively simple set is simple. The simple set S constructed by Post in [5] is effectively simple. This latter is no accident. In fact it is not unreasonable to claim that any direct attack on the problem of constructing a simple set must result in an effectively simple set. Our purpose here is to obtain a simple set which is not effectively simple. We will make strong use of the recursion theorem of Kleene [2]; however, we will use it in the informal manner of Myhill [4]. Our notation is that of [2].

We introduce a recursive function E:

$$E(0) = \mu x T_1((x)_0, (x)_1, (x)_2);$$
$$E(s+1) = \mu x[x > E(s) \,\&\, T_1((x)_0, (x)_1, (x)_2)].$$

We will need E to simultaneously enumerate all the recursively enumerable sets in a fashion suitable for the proving of our theorem. It is a peculiarity of our proof that we cannot rely merely on the usual properties associated with any standard enumeration of the recursively enumerable sets; instead, we are forced to specify a particular enumeration. For each e and s we define a finite set W_e^s: for each m, $m \in W_e^s$ if and only if for some $i \leq s$,

$$m = U((E(i))_2) \,\&\, e = (E(i))_0.$$

Then for each e, $W_e^0 \subseteq W_e^1 \subseteq W_e^2 \subseteq \cdots$, and $W_e = \bigcup \{W_e^s \mid s \geq 0\}$. We

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[2] Smullyan actually requires that f be recursive, but it is easy to show the two definitions equivalent.

say s defines $f_e(n)$ if $e=(E(s))_0$ and $n=(E(s))_1$. If s defines $f_e(n)$, then $f_e(n)=U((E(s))_2)$. For each e and n, let

$$S(e, n) \simeq \mu s \quad (s \text{ defines } f_e(n)).$$

S is partial recursive, and $S(e, n)$ is defined if and only if $f_e(n)$ is defined.

Let 0 denote the empty set. It is clear there exists a recursive function g such that for each e, i and z, we have

$$W_{g(e,i,z)} = \begin{cases} \{2^i \cdot 3^t \mid E(S(e,z)) < t \leqq E(S(e,z)) + f_e(z)\} & \text{if } f_e(z) \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

The recursion theorem tells us that there exists a recursive function z such that for each e and i, we have

$$W_{z(e,i)} = W_{g(e,i,z(e,i))} = \begin{cases} \{2^i \cdot 3^t \mid E(S(e, z(e,i))) < t \leqq E(S(e, z(e,i))) + f_e(z(e,i))\} \\ \qquad\qquad \text{if } f_e(z(e, i)) \text{ is defined,} \\ 0 \qquad \text{otherwise.} \end{cases}$$

We note some properties of z:

(1) if $f_e(z(e, i))$ is defined, then $f_e(z(e, i))$ is equal to the cardinality of $W_{z(e,i)}$;

(2) if $i \neq j$, then $W_{z(e,i)} \cap W_{z(e,j)} = 0$;

(3) if $f_e(z(e, i))$ is defined, then for all n, $W_n^{S(e,z(e,i))} \cap W_{z(e,i)} = 0$;

(4) if $i \neq j$ and both $f_e(z(e, i))$ and $f_e(z(e, j))$ are defined, then $z(e, i) \neq z(e, j)$.

To prove (3), let $s = S(e, z(e, i))$ and let $m \in W_n^s \cap W_{z(e,i)}$. Then $m \leqq E(s)$, since $m = U((E(i))_2)$ for some $i \leqq s$, and since E is an increasing function. (Recall that $U(x) \leqq x$ for all x.) But $m > E(s)$, since $m = 2^i \cdot 3^t$ for some $t > E(s)$.

THEOREM 1. *There exists a simple set which is not effectively simple.*

PROOF. We will define a sequence A, B, Q_0, Q_1, Q_2, $\cdots$ of simultaneously recursively enumerable sets. A will be simple, but not effectively simple. B will be such that if $e \in B$, then $W_e \cap A \neq 0$. We will see to it that if W_e is infinite, then $e \in B$. Each Q_e will be finite and will contain a set that will serve as a witness to the fact that f_e does not effectively bound the cardinalities of the finite subsets of the complement of A.

Stage $s=0$. We set $A^0 = B^0 = Q_i^0 = 0$ for all i.

Stage $s>0$. Let $e=(E(s))_0$ and $n=(E(s))_1$. Thus s defines $f_e(n)$. We perform the following two operations in the indicated order:

(a) We set $Q_j^s=Q_j^{s-1}$ for all $j\neq e$. If there is no i such that $i\leqq e$ and $n=z(e,\ i)$, we set $Q_e^s=Q_e^{s-1}$. If there is such an i, then by (4) it is unique. In addition, $S(e,\ z(e,\ i))$ is defined and

$$W_{z(e,i)} = \{2^i\cdot 3^t \mid E(S(e, z(e, i))) < t \leqq E(S(e, z(e, i))) + f_e(z(e, i))\}.$$

We set $Q_e^s=Q_e^{s-1}\cup W_{z(e,i)}$.

(b) If $e\in B^{s-1}$ or if there is no m such that

$$m \in W_e^s \;\&\; (j)_{j\leq e}\,(m \notin Q_j^s),$$

then we set $B^s=B^{s-1}$ and $A^s=A^{s-1}$. If $e\notin B^{s-1}$ and there is an m with the above property, let i be the least one. We set $B^s=B^{s-1}\cup\{e\}$ and $A^s=A^{s-1}\cup\{i\}$.

Let $A=\bigcup\{A^s\mid s\geqq 0\}$ and $B=\bigcup\{B^s\mid s\geqq 0\}$. Since E and z are recursive, it follows A is recursively enumerable. For each e, let $Q_e=\bigcup\{Q_e^s\mid s\geqq 0\}$. Q_e is finite; in fact,

$$Q_e = \bigcup\{W_{z(e,i)} \mid i \leqq e\},$$

since $Q_e^{s-1}\neq Q_e^s=Q_e^{s-1}\cup W_{z(e,i)}$ if and only if $i\leqq e$ and $s=S(e,\ z(e,\ i))$.

LEMMA 1. *If W_e is infinite, then $A\cap W_e\neq 0$.*

PROOF. We know Q_j is finite for every j. Let m be a member of W^e which is greater than every member of Q_j for all $j\leqq e$. Let s be such that $m\in W_e^s$. First we suppose $e\in B^{s-1}$. Then there must be a $t<s$ such that $e\notin B^{t-1}$ and $e\in B^t$. At stage t we must have performed operation (b) in such a manner that $B^t=B^{t-1}\cup\{e\}$ and $A^t=A^{t-1}\cup\{i\}$, where $i\in W_e^s$. Now we suppose $e\notin B^{s-1}$. We have

$$m \in W_e^s \;\&\; (j)_{j\leq e}\,(m \notin Q_j^s).$$

But then operation (b) at stage s forces us to put a member of W_e^s in A^s.

LEMMA 2. *If $m\in W_e^s-Q_j^s$, then $m\notin Q_j$.*

PROOF. Suppose for the sake of a reductio ad absurdum that $m\in W_e^s-Q_e^s$ and $m\in Q_j$. Since $Q_j=\bigcup\{W_{z(j,i)}\mid i\leqq j\}$, there must be an $i\leqq j$ such that $m\in W_{z(j,i)}$. Since $W_{z(j,i)}$ is nonempty, $f_j(z(j,\ i))$ is defined. Let $t=S(j,\ z(j,\ i))$. Then t defines $f_j(z(j,\ i))$, and consequently,

$$Q_j^t = Q_j^{t-1} \cup W_{z(j,i)},$$

since $i \leqq j$. Since $m \notin Q_j^s$, we must have $s < t$. Since $m \in W_e^s$, we have

$$m \in W_e^t \cap W_{z(j,i)} \neq 0.$$

But this last contradicts (3).

LEMMA 3. *If $m \in Q_i \cap A$, then there exists an s and an e such that $(E(s))_0 = e < i$ and $\{e\} = B^s - B^{s-1}$ and $\{m\} = A^s - A^{s-1}$.*

PROOF. Since $m \in A$, there is an s such that $\{m\} = A^s - A^{s-1}$. Let $e = (E(s))_0$. Since $A^s \neq A^{s-1}$, we must have $\{e\} = B^s - B^{s-1}$. In addition,

$$m \in W_e^s \;\&\; (j)_{j \leqq e}\, (m \notin Q_j^s).$$

It follows from Lemma 2 that $(j)_{j \leqq e}\,(m \notin Q_j)$. But then $e < i$, since $m \in Q_i$.

LEMMA 4. *The set $Q_i \cap A$ has at most i members.*

PROOF. Suppose m and n are distinct members of $Q_i \cap A$. Lemma 3 guarantees the existence of $s(m)$, $e(m)$, $s(n)$ and $e(n)$ with properties as stated in the conclusion of Lemma 3. Thus

$$\{m\} = A^{s(m)} - A^{s(m)-1} \;\&\; \{n\} = A^{s(n)} - A^{s(n)-1}.$$

Since $m \neq n$, it follows $s(m) \neq s(n)$. But then $e(m) \neq e(n)$, since

$$\{e(m)\} = B^{s(m)} - B^{s(m)-1} \;\&\; \{e(n)\} = B^{s(n)} - B^{s(n)-1}.$$

We also know from Lemma 3 that $e(m) < i$ and $e(n) < i$. Thus we can map the set $Q_i \cap A$ in a one-to-one fashion into the set $\{e \mid e < i\}$.

LEMMA 5. *For each e, there is a z such that W_z is contained in the complement of A and such that either $f_e(z)$ is undefined or $f_e(z)$ is not greater than the cardinality of W_z.*

PROOF. Fix e. We show that some member of the sequence, $z(e, 0)$, $z(e, 1), \cdots, z(e, e)$ serves as the desired z. Suppose there is an $i \leqq e$ such that $f_e(z(e, i))$ is undefined. Then $W_{z(e,i)} = 0$, and the lemma is proved. Suppose then that $f_e(z(e, i))$ is defined for all $i \leqq e$. By (1), $f_e(z(e, i))$ is not greater than the cardinality of $W_{z(e,i)}$ for any $i \leqq e$. Thus it suffices to find an $i \leqq e$ such that $W_{z(e,i)} \cap A = 0$. The sets,

$$W_{z(e,0)}, W_{z(e,1)}, \cdots, W_{z(e,e)},$$

are nonempty and disjoint. If each of them has a member in A, then their union has at least $e+1$ members in A. But their union is Q_e, and Lemma 4 tells us that Q_e has at most e members in A.

It follows from Lemma 5 that A is not effectively simple. It also follows from Lemma 5 that the complement of A is infinite, since otherwise, the constant function

$$f(n) = 1 + \text{cardinality of the complement of } A$$

would constitute a counterexample to Lemma 5. Finally, by Lemma 1, A is simple.

Post [5] calls a recursively enumerable set W hyper-simple if its complement is infinite, and if there does not exist a recursively enumerable sequence of disjoint, finite sets, each one of which contains a member of the complement of W. It can be shown with the help of Lemma 4 that A is not hyper-simple.

The proof of Theorem 1 above is, as far as we know, the first proof in recursion theory to make simultaneous use of the recursion theorem and the priority method of Friedberg [1] and Muchnik [3]. The priority method was needed to resolve the inevitable conflict between putting elements in A as required by Lemma 1 and keeping them out of A as required by Lemma 4. Thus in operation (b), we are not allowed to take m from W_e^s and add it to A^s if for some $j \leqq e$, $m \in Q_j^s$. The recursion theorem was needed to prove that our system of priorities does eventually resolve all conflicts happily; in particular, the recursion theorem made possible the proof of Lemma 2.

References

1. R. M. Friedberg, *Two recursively enumerable sets of incomparable degrees of unsolvability*, Proc. Nat. Acad. Sci. U. S. A. **43** (1957), 236–238.

2. S. C. Kleene, *Introduction to metamathematics*, Van Nostrand, New York, 1952.

3. A. A. Muchnik, *Negative answer to the problem of reducibility of the theory of algorithms*, Dokl. Akad. Nauk SSSR **108** (1956), 194–197. (Russian)

4. John Myhill, *Creative sets*, Z. Math. Logik Grundlagen Math. **1** (1955), 97–108.

5. E. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc. **50** (1944), 284–316.

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