# UNIQUE FACTORIZATION OF IDEALS INTO NONFACTORABLE IDEALS

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The purpose of this note is to prove a theorem which shows a connection between the definition of a prime ideal in classical algebraic number theory and the usual definition of a prime ideal. A proper ideal in an integral domain with unit element is an ideal different from the unit ideal and the zero ideal. An ideal $A$ will be called nonfactorable provided $A$ is a proper ideal and $A = BC$ (where $B$ and $C$ are ideals) implies that either $B$ or $C$ is the unit ideal.

THEOREM. *If $J$ is an integral domain with unit such that every proper ideal of $J$ is either a nonfactorable ideal or can be factored uniquely into a product of nonfactorable ideals, then $J$ is a Dedekind domain and the nonfactorable ideals are prime in the usual sense.*

PROOF. Let $P$ be a proper prime ideal of $J$ and $p \neq 0$ be an element of $P$. There exist nonfactorable ideals $N_1, \cdots, N_n$ in $J$ such that $(p) = N_1 \cdot \cdots \cdot N_n$. Let $N_i$ be an arbitrary member of the collection $N_1, \cdots, N_n$. Since $(p)$ is an invertible ideal, it follows that $N_i$ is an invertible ideal and consequently $N_i$ is finitely generated (see [1, p. 272]).

Let $x$ be any element of $J$ such that $x$ is not an element of $N_i$. Since $N_i$ is finitely generated, then $N_i + (x)$ is finitely generated. The cancellation law for ideals is valid in $J$ (an obvious consequence of the unique factorization property) and therefore finitely generated ideals are invertible (see [2, p. 13]). Hence $N_i + (x)$ is invertible and since $N_i + (x) \supset N_i$ there exists an ideal $Q$ in $J$ such that $[N_i + (x)] \cdot Q = N_i$. It is clear that $N_i + (x) = J$ and $N_i$ is a maximal ideal.

Since $P$ is a prime ideal and $P \supset N_1 \cdot \cdots \cdot N_n$ it follows that $P \supset N_i$ for some $i$ and therefore $P$ is invertible. Hence every proper prime ideal of $J$ is invertible and $J$ is a Dedekind domain (see [3, Theorem 7, p. 33]).

## REFERENCES

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