A SIMPLY CONNECTED 3-MANIFOLD IS S^3 IF IT IS THE SUM OF A SOLID TORUS AND THE COMPLEMENT OF A TORUS KNOT

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It has been shown [6] that any closed, connected, orientable 3-manifold can be constructed by removing a finite number, k, of mutually exclusive tame solid tori[1] from the 3-sphere, S^3, and then sewing them back in some possibly different manner. In particular, any homotopy 3-sphere can be obtained in this manner; thus it gives an approach to the Poincaré Conjecture.

We wish to consider the case $k=1$. To this end let T be a tame solid torus in S^3 and let M be a simply connected 3-manifold containing a solid torus S such that there is a homeomorphism h of $S^3-\mathrm{Int}(T)$ onto $M-\mathrm{Int}(S)$. It is known [1] that M is S^3 if T is unknotted or if T has the type of a trefoil knot. In [5] it is asserted that M is S^3 in every case. The proof given in [5] shows only that h must take some longitude of T onto a longitude of S—a fact that is not sufficient to prove this assertion. The purpose of this paper is to provide a proof that M is S^3 in the case that T has the type of a torus knot.

We first give a canonical description of the torus knot of type (p, q). We consider S^3 as the one point compactification of E^3 which we represent in cylindrical coordinates (r, θ, z). Let R be the torus $R=\{(r, \theta, z): (r-2)^2+z^2=1\}$. Let p and q be relatively prime positive integers. We assume, for convenience, that $p>q$; since it is known that the torus knots of types (p, q) and (q, p) are equivalent. For $1\leqq k\leqq p$ let $x_k=(3, 2k\pi/p, 0)$ and $y_k=(1, 2k\pi/p, 0)$. Let a_k be an arc on R from x_k to y_{k+q} that lies in the positive z half space and which increases monotonically with θ. (It is to be understood that the subscripts are to be reduced modulo p.) Let b_k be the arc on R from x_{k+q} to y_{k+q} that lies in the negative z half space and the plane $\theta=2(k+q)\pi/p$. We denote the simple closed curve $\bigcup_{k=1}^{p}(a_k\cup b_k)$ by $K_{p,q}$. Figure 1 shows $K_{5,3}$.

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[1] A solid torus T is a set homeomorphic to the Cartesian product of a disk and a simple closed curve. Any simple closed curve on Bd T which bounds a disk in T but which does not separate Bd T is called a meridian of T. Any simple closed curve on Bd T which does not separate Bd T, which is not a meridian of T, and which intersects some meridian of T in exactly one point is called a longitude of T.

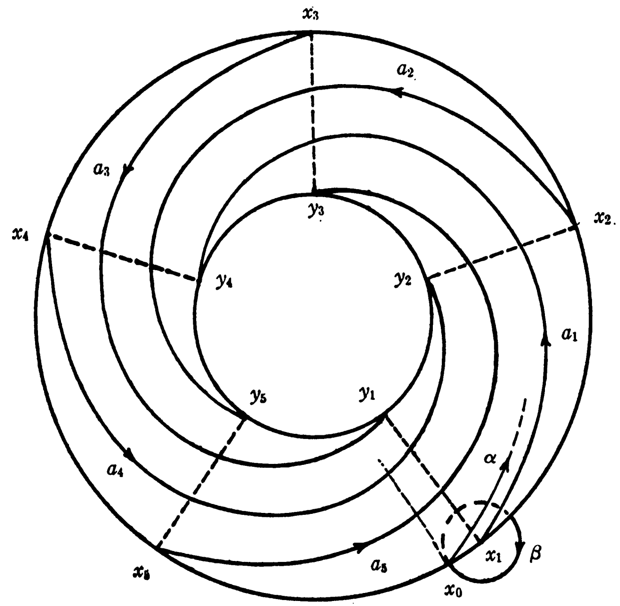

FIGURE 1

It is well known that any torus knot (i.e., a simple closed curve which lies on a tame, unknotted torus in S^3) can be taken onto one of the curves $K_{p,q}$ by a (possibly orientation reversing) homeomorphism of S^3 onto itself.

The fundamental group of $S^3-K_{p,q}$ can be given a presentation in the standard manner (see [4, Chapter III]) with the overcrossings, a_k, representing generators and the undercrossings, b_k, giving the relations. The convention we adopt is that the generator a_k represents a loop which begins at the base point x_0 (which we choose to be obtained by moving x_1 slightly in the negative θ direction as shown in Figure 1), passes under the segment a_k from left to right (in terms of the orientation assigned to a_k), and returns to x_0. It is understood that this loop passes under no other segment in the projection of the knot.

The relation corresponding to the undercrossing b_k may be read in the standard manner (see Figure 2), and we obtain the following presentation of $\pi_1(S^3-K_{p,q})$.

$$\{a_1, a_2, \cdots, a_p : a_k a_{k+1} \cdots a_{k+q-1} = a_{k+1} a_{k+2} \cdots a_{k+q}\}, \qquad k = 1, 2, \cdots, p. \tag{1}$$

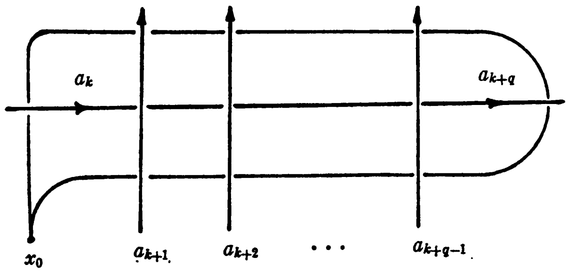

FIGURE 2

If we let $X=a_1a_2 \cdots a_q$, $Y=a_1a_2 \cdots a_p$, we show in the following manner that X and Y generate $\pi_1(S^3-K_{p,q})$. Since p and q are relatively prime, there are integers r and s such that $rp+sq=1$. If $r>0$ (hence $s<0$), then

$$\begin{aligned} Y^r &= (a_1a_2 \cdots a_p)(a_1a_2 \cdots a_p) \cdots (a_1a_2 \cdots a_p) \\ &= a_1(a_2a_3 \cdots a_{q+1})(a_{q+2}a_{q+3} \cdots a_{2q+1}) \cdots (a_{p-q+1}a_{p-q+2} \cdots a_p) \\ &= a_1X^{-s}. \end{aligned}$$

While if $r<0$ (hence $s>0$), then

$$\begin{aligned} X^s &= (a_1a_2 \cdots a_q)(a_{q+1}a_{q+2} \cdots a_{2q}) \cdots (a_{p-q+2}a_{p-q+3} \cdots a_pa_1) \\ &= (a_1a_2 \cdots a_p)(a_1a_2 \cdots a_p) \cdots (a_1a_2 \cdots a_p)a_1 \\ &= Y^{-r}a_1. \end{aligned}$$

In either case we see that $a_1=Y^rX^s$. By the same technique we can show that $a_2=X^{-s}Y^rX^{2s}, \cdots, a_k=X^{(1-k)s}Y^rX^{ks}$. Using these relationships it can be shown that $\pi_1(S^3-K_{pq})$ has the following presentation in terms of the elements X and Y:

$$\{X, Y: X^p = Y^q\}. \tag{2}$$

To verify this one need only show that the maps ϕ: $(1)\rightleftarrows(2):\psi$ generated by $\phi(a_k) = X^{(1-k)s}Y^rX^{ks}$, $\psi(X) = (a_1a_2 \cdots a_q)$, and $\psi(Y) = (a_1a_2 \cdots a_p)$ preserve relations and that $\phi\psi=1$ and $\psi\phi=1$.

We are now prepared to prove our result. First we state the following well-known

LEMMA. *Suppose M and N are 3-manifolds containing solid tori T and S, respectively, and h is a homeomorphism of $M-\mathrm{Int}(T)$ onto $N-\mathrm{Int}(S)$ which takes* Bd T *onto* Bd S *and which takes some meridian of T onto a meridian of S. Then h can be extended to a homeomorphism of M onto N.*

THEOREM. *Suppose T is a tubular neighborhood of the torus knot $K_{p,q}$, M is a 3-manifold, and S is a solid torus in M such that $S^3-\text{Int}(T)$ is homeomorphic to $M-\text{Int}(S)$. Then if M is simply connected, M is homeomorphic to S^3.*

PROOF. We assume without loss of generality that $T\cap R$ is an annular neighborhood of $K_{p,q}$ on R one of whose boundary components, which we denote by α, contains the point x_0. By choosing the proper orientation for α we see that α represents the element $(a_1a_2 \cdots a_p)^q=Y^q$ of $\pi_1(S^3-\text{Int}(T))$. We let β be the component of the intersection of the plane $z=0$ with Bd T which contains x_0. Properly oriented, β represents the element a_1 of $\pi_1(S^3-\text{Int}(T))$; furthermore α and β serve as generators for the abelian group $\pi_1(\text{Bd } T)$.

Now let $h: S^3-\text{Int}(T)\to M-\text{Int}(S)$ be the homeomorphism given by the hypothesis. Let γ be a meridian of S containing the point $h(x_0)$. Now $h^{-1}(\gamma)$ represents an element of the form $\alpha^m\beta^n = (a_1a_2 \cdots a_p)^{qm}a_1^n = Y^{qm}(Y^rX^s)^n$. We obtain $\pi_1(M)$ from $\pi_1(S^3-\text{Int}(T))$ by adjoining the additional relation $h^{-1}(\gamma)=1$. This yields the following:

$$\pi_1(M) = \{X,\ Y: X^p = Y^q,\ Y^{qm}(Y^rX^s)^n = 1\}. \tag{3}$$

We will show that $m=0$, $n=\pm 1$; hence that $h^{-1}(\gamma)$ is a meridian of T. In light of the previous lemma, this will complete the proof.

The commutator quotient group of $\pi_1(M)$ has the presentation $\{a_1: a_1^{pqm+n}=1\}$. Since this group is trivial, we must have $n=\pm 1 -pqm$. Now neither p nor q is 1; for otherwise T would be unknotted. Suppose that $m\neq 0$. Then $n\neq 0$, ± 1. We will show that this assumption leads to a contradiction as follows.

Consider the group:

$$G_{p,q,n} = \{R_1, R_2, R_3: R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^q = (R_2R_3)^p = (R_1R_3)^n = 1\}.$$

If p, q, $n\neq 0$, ± 1, this group is nontrivial [2, p. 55]. $G_{p,q,n}$ may be represented as follows. Let PQN be a triangle with angles $\pi/|p|$, $\pi/|q|$, and $\pi/|n|$ in the hyperbolic plane, the plane, or the 2-sphere according as $\pi/|p|+\pi/|q|+\pi/|n|$ is less than, equal to, or greater than π. We let R_1, R_2, and R_3 be reflections through the "lines" on QN, PQ, and PN, respectively. Then R_1R_2, R_2R_3, and R_1R_3, respectively, are rotations of angles $2\pi/|q|$, $2\pi/|p|$, and $2\pi/|n|$ about the vertices Q, P, and N. We can obtain a nontrivial representation of $\pi_1(M)$ into $G_{p,q,n}$ given by $\eta(X)=(R_2R_3)^q$, $\eta(Y)=(R_1R_2)^p$. Note that $\eta(X^p) = (R_2R_3)^{pq} = 1 = (R_1R_2)^{pq} = \eta(Y^q)$, and that $\eta(Y^{qm}(Y^rX^s)^n)$

$= (R_1R_2)^{pqm}((R_1R_2)^{pr}(R_2R_3)^{qs})^n = ((R_1R_2)^{1-qs}(R_2R_3)^{1-pr})^n = (R_1R_2R_2R_3)^n$ $= (R_1R_3)^n = 1$. Hence η is a homomorphism. This gives the contradiction that $\pi_1(M)$ is nontrivial. Thus we have $m=0$. Since γ is a simple closed curve, we must have $n = \pm 1$, and the proof is complete.

Note that the homeomorphism between S^3 and M given by the conclusion of the above theorem was an extension of the homeomorphism between $S^3 - \mathrm{Int}(T)$ and $M - \mathrm{Int}(S)$ given by the hypothesis. From this observation we can state the following as a direct consequence of the above theorem. (It was pointed out by the referee that this corollary is a known result. See [**3**, p. 183].)

COROLLARY. *Suppose T is a tubular neighborhood of the torus knot $K_{p,q}$ and h is a homeomorphism of $S^3 - \mathrm{Int}(T)$ onto a tame subset of S^3. Then h can be extended to a homeomorphism of S^3 onto itself.*

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