LINDELÖFIAN MEROMORPHIC FUNCTIONS[1]

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I. **Introduction.** The principal object of this paper is to establish a result (Theorem 1) on Lindelöfian meromorphic functions which has as an immediate consequence the known [**4**, p. 442] relation $O_{HB} \subset O_L$ in the classification theory of Riemann surfaces. In fact, Theorem 1 emerged out of an attempt to obtain a proof, presumably simpler and more direct than the known (cf. [**4**, p. 442] and [**6**, p. 105]), of this inclusion.

II. **Preliminaries.** In this section we provide some needed background material. Details and proofs of propositions concerning Lindelöfian maps can be found in [**4**, pp. 424–430]. A proof of proposition 4 can be found in [**1**, pp. 210–211].

Let W and R be Riemann surfaces and $f\colon W \to R$ be a (complex) analytic map. f is said to be Lindelöfian if it is of bounded characteristic, (see [**4**]). The following proposition is implicit in [**4**].

PROPOSITION 1. *There is no nonconstant Lindelöfian map with domain a parabolic Riemann surface.*

Henceforth W stands for a hyperbolic Riemann surface and all analytic maps to be considered will be nonconstant. For any hyperbolic surface F, $g_F(\cdot, q)$ denotes the Green's function of F with pole at q. The following criterion [**4**] holds for Lindelöfian maps.

PROPOSITION 2. *An analytic map $f\colon W \to R$ is Lindelöfian if and only if, for every $r \in R$ and $q \in W$ with $f(q) \neq r$,*

$$G(f, q, r) \equiv \sum_{f(p)=r} n(p, f) g_W(p, q) < \infty, \tag{1}$$

where $n(p, f)$ is the multiplicity of f at p.

A Lindelöfian meromorphic function is a Lindelöfian map with ranges the extended plane. Hereafter, we shall denote this class of functions by L. For this class of functions the following characterization [**4**; **7**; **8**] holds.

Received by the editors September 24, 1962.

[1] The contents of this paper formed part of a Ph.D. thesis submitted in April, 1962 to the Graduate Division, University of California, Los Angeles. The author is greatly indebted to his supervisor, Professor L. Sario, for encouragement and advice.

[2] This work has been supported by the U. S. Army Research Office (Durham) Grant No. DA-ARO(D)-31-124-G40, University of California, Los Angeles.

PROPOSITION 3. *A meromorphic function f on W is in L if and only if, for $q \in W$,*

$$\log |f(q)| = G(f, q, \infty) - G(f, q, 0) + P(q) - P'(q) \tag{2}$$

where P and P' are nonnegative harmonic functions on W.

It is immediate from Propositions 2 and 3 that a meromorphic f on W is in L if and only if, for every complex number a and $q \in W$,

$$\log |f(q) - a| = G(f, q, \infty) - G(f, q, a) + P_a(q) - P_a'(q), \tag{2a}$$

where P_a and P_a' are nonnegative harmonic functions on W. Further, it is clear from Proposition 3 that a meromorphic function on the unit disc is in L if and only if it is of bounded type (see [5, p. 188]).

We shall conclude this section by stating some definitions and a result due to Parreau. For any Riemann surface F let $\mathcal{P}(F)$ denote the class of nonnegative harmonic functions on F. A function h in $\mathcal{P}(F)$ is said to be quasi-bounded if it is the limit of a nondecreasing sequence of bounded members of $\mathcal{P}(F)$. s in $\mathcal{P}(F)$ is said to be singular if the only bounded member of $\mathcal{P}(F)$ majorized by s is zero. From these definitions it follows that the only quasi-bounded function majorized by a singular function is zero. Also, the following decomposition theorem holds.

PROPOSITION 4. *Any h in $\mathcal{P}(F)$ can be written in a unique way in the form*

$$h = h_B + h_S, \tag{3}$$

where h_B is quasi-bounded and h_S is singular. Further, $h_B = \lim_{n\to\infty}$ G.H.M. min (h, n) where G.H.M. stands for "the greatest harmonic minorant of."

Note. h_B in (3) is called the quasi-bounded part of h and h_S the singular part.

III. **Main result.** We shall now state

THEOREM 1. *Let $f \in L$ on W. Then there exists at most one complex number a such that, in* (2a), *the difference between the quasi-bounded parts of P_a and P_a' is a constant.*

The proof of this theorem is by uniformization. Let (U, π) be a universal cover of W, U being realized as the unit disc of the complex plane. Let $\mathfrak{J}$ be the group of conformal cover transformations of U relative to π. For any map ϕ with domain W let $\phi^* = \phi \circ \pi$ be the lifting to U.

We shall now establish the following simple lemmas.

LEMMA 1. *If h in $\mathcal{P}(U)$ is automorphic relative to $\mathfrak{Z}$, then so is its quasi-bounded part h_B.*

PROOF. By Proposition 4, $h_B=\lim_{n\to\infty} h_n$, $h_n=$G.H.M. min (h, n). Hence it is enough to show that h_n is automorphic relative to $\mathfrak{Z}$. Let $z\in U$, $\tau\in\mathfrak{Z}$. Then $h_n(\tau z)\leqq h(\tau z)=h(z)$ and $h_n(\tau z)\leqq n$. Hence, by the definition of G.H.M., $h_n(\tau z)\leqq h_n(z)$. Since this holds for every $z\in U$ and every $\tau\in\mathfrak{Z}$, a group, we obtain the reverse inequality, thus completing the proof.

LEMMA 2. *If h in $\mathcal{P}(U)$ is quasi-bounded and automorphic relative to $\mathfrak{Z}$, then the harmonic function h', defined on W by*

$$h'(p) = h(z), \qquad \pi(z) = p,$$

is quasi-bounded.

PROOF. By the proof of Lemma 1, $h = \lim_{n\to\infty} h_n$, where h_n is bounded, automorphic relative to $\mathfrak{Z}$, and increases with n. Hence $h'=\lim_{n\to\infty} h_n'$, where h_n' is bounded and increases with n. The result now follows by the definition of a quasi-bounded function.

LEMMA 3. *s in $\mathcal{P}(W)$ is singular, if and only if s^* is.*

PROOF. Suppose that s is singular. By Lemma 1, s_B^*, the quasi-bounded part of s^* is automorphic relative to $\mathfrak{Z}$. Hence, by Lemma 2, $s_B^{*\prime}$ defined on W by

$$s_B^{*\prime}(p) = s_B^*(z), \qquad \pi(z) = p,$$

is quasi-bounded on W, and is majorized by the singular s. So $s_B^{*\prime}$ and hence s_B^* is zero, that is, s^* is singular.

We omit the easy proof of the converse.

LEMMA 4. *s in $\mathcal{P}(U)$ is singular if and only if it has radial limit zero almost everywhere, i.e.,* $\lim_{r\to 1} s(re^{it})=0$ *except for a set of t of Lebesgue measure zero.*

This result is known. For a proof, see, for instance, [**2**, p. 540].

For the proofs of the following three lemmas we refer to [**5**, pp. 214, 207, and 209].

LEMMA 5. $g_W(\pi(z), q)=G(\pi, z, q)$.

LEMMA 6. *The sum of a convergent series of Green's functions of U has radial limit zero almost everywhere.*

LEMMA 7. *If $\phi\in L$ on U, then* $\lim_{r\to 1}\phi(re^{it})$ *exists for almost all t, and the set of these radial limits corresponding to a set of t of positive Lebesgue measure contains more than two points.*

PROOF OF THEOREM 1. Suppose the conclusion of the theorem is false. Then there exist complex numbers a_1, a_2, $a_1 \neq a_2$, such that, for $i=1, 2$,

$$\log |f(p) - a_i| = G(f, p, \infty) - G(f, p, a_i) + K_i + s_i(p) - s_i'(p),$$

where s_i and s_i' are singular members of $\mathcal{P}(W)$ and the K_i are constants. It now follows, from Lemmas 3 and 5, that,

$$\log |f^*(z) - a_i| = G(f^*, z, \infty) - G(f^*, z, a_i) + K_i + S_i(z) - S_i'(z), \quad i = 1, 2, \tag{4}$$

where S_i and S_i' are singular members of $\mathcal{P}(U)$. (4), together with Proposition 3, shows that $f^*(z) - a_i$ is Lindelöfian on U, and hence, by Lemma 7, f^* has radial limits almost everywhere. Also, (4), together with Lemmas 4 and 6, yields that these radial limits lie, for almost all points of the unit circle, on both the circles,

$$|z - a_i| = e^{K_i}, \qquad i = 1, 2.$$

Since $a_i \neq a_2$, this is a contradiction to elementary geometry in view of the second part of Lemma 7. Hence, the theorem is true.

REMARK. There can be an exceptional a in Theorem 1. For instance, for the identity map of the unit disc, 0 is exceptional. More generally, quotients of functions of Seidel's class (U) (see [6, p. 32]) have zero as exceptional value.

Let HB denote the class of bounded harmonic functions and O_{HB} the class of Riemann surfaces which do not admit nonconstant members of the class HB. Let O_L have a similar meaning. Then we have

COROLLARY 1. $O_{HB} \subset O_L$.

PROOF. This is immediate from Proposition 1, Theorem 1 and the definition of a quasi-bounded function.

COROLLARY 2. *If there exists a Lindelöfian map f from W to a compact Riemann surface R, then $W \notin O_{HB}$.*

PROOF. Let g be a nonconstant meromorphic function on R so that g assumes every value only a finite number of times. Since f is Lindelöfian, this implies, in view of Proposition 2, that $h = g \circ f \in L$ on W. The result now follows from Corollary 1.

REMARK. Actually it is proved in [4] that, in Corollary 2, "compact" can be replaced by "parabolic." But we have been unable to prove this stronger result by our methods.

IV. **The classes O_{HD} and O_L.** Let HD be the class of harmonic functions with a finite Dirichlet integral on a Riemann surface and AB the class of bounded analytic functions. Denote by L' the class of those members of L which are pole-free. It is known [4, p. 442] that $O_L \subset O_{L'} \subset O_{AB}$. Making use of known examples we shall now establish

THEOREM 2. *There is no inclusion relation between O_{HD} and O_L, nor between O_{HD} and $O_{L'}$.*

PROOF. $O_{HD} \not\subset O_{L'}$. For, otherwise, $O_{HD} \subset O_{AB}$ and this is known to be false [1, p. 264, Theorem 26H]. Hence, it is enough to show that $O_L \not\subset O_{HD}$. To do this we shall make use of the "ends" considered in [3]. Consider a Riemann surface F with the following properties:

(a) the surface is parabolic,

(b) the complement of every compact subset of F has exactly one component whose closure is not compact.

An "end" is a subregion of a surface of the above type whose complement is compact. It is clear that any harmonic function, bounded on the closure of an end Ω with smooth boundary, must belong to the class HD on Ω.

Heins [4, p. 442] has established that there exists an end whose closure admits a nonconstant bounded harmonic function but no subend of which admits a nonconstant function of class L. This result, together with the preceding remark, yields that $O_L \not\subset O_{HD}$, thus completing the proof.

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