THE u-ALGEBRA OF A RESTRICTED LIE ALGEBRA IS FROBENIUS

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Let C be a field of characteristic $p \neq 0$, L a restricted finite dimensional Lie algebra over C and $u(L)$ its restricted enveloping algebra or u-algebra [3, p. 192]. If $a_1, a_2, \cdots, a_n$ is an ordered basis for L, a basis for $u(L)$ is given by the monomials $a^I = a_1^{i_1} \cdots a_n^{i_n}$, $0 \leqq i_s < p$ [3, p. 190].

Let S be the C-subspace of $u(L)$ spanned by all these monomials except $a_1^{p-1} \cdots a_n^{p-1}$. Then the usual straightening procedure, working both in the universal enveloping algebra and in $u(L)$ and using both commutation and the p-power map, can easily be used (cf. [3, Chapter V, especially Lemma 4, p. 189]) to prove

LEMMA. *Let $a^I = a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}$, $a^J = a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n}$. If $\sum_s i_s + \sum_s j_s < n(p-1)$ then $a^I a^J$ is in S. If $\sum_s i_s + \sum_s j_s = n(p-1)$ then $a^I a^J$ is in S also unless for each s, $1 \leqq s \leqq n$, $i_s + j_s = p-1$, in which case $a^I a^J \equiv a_1^{p-1} a_2^{p-1} \cdots a_n^{p-1}$ modulo S.*

An associative algebra A is called a Frobenius algebra if there is a left A-module isomorphism

$$A \cong \mathrm{Hom}_C (A, C) = A^*,$$

where A^* is a left A-module via $(a\phi)(x) = \phi(xa)$ for ϕ in A^* and a, x in A [2, p. 3].

THEOREM. *Let L be a restricted finite dimensional Lie algebra over a field C of characteristic p. Let $u(L)$ be the u-algebra of L. Then $u(L)$ is a Frobenius algebra.*

PROOF. Let ϕ in $\mathrm{Hom}_C(u(L), C) = u(L)^*$ be given by

$$\phi(S) = 0, \qquad \phi(a_1^{p-1} a_2^{p-1} \cdots a_n^{p-1}) = 1.$$

Let x be in $u(L)$ and suppose $x\phi = 0$. Using the basis described above, $x = \sum c_I a^I$ with c_I in C. As usual we define degree a^I = degree $a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}$ as $\sum_s i_s$ and degree x as the maximum of the degrees of the a_I with $c_I \neq 0$. If degree $x = k$ and $c_H \neq 0$ for degree $a^H = k$, we let

$$a^{H'} = a_1^{p-1-h_1} a_2^{p-1-h_2} \cdots a_n^{p-1-h_n}.$$

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Then

$$0 = x\phi(a^{H'}) = \phi(a^{H'}x) = c_H$$

by the lemma. Thus $x=0$.

Clearly the map $x \rightarrow x\phi$ is a $u(L)$ monomorphism of $u(L)$ into $u(L)^*$. Since both are finite dimensional vector spaces of the same dimension over C, this map must be an isomorphism.

REMARKS.

1. It is easily seen by means of examples that the theorem cannot be strengthened to show that $u(L)$ is symmetric, i.e., that the isomorphism between $u(L)$ and $u(L)^*$ can be chosen to be a two-sided module map. For example, if C is an imperfect field of characteristic two, β in C but not in C^2, L the C-derivation algebra of $C(\sqrt{\beta})$, then $u(L)$ is not symmetric.

2. Let hd_R denote the left homological dimension of the R-module M. It is known that if R is a Frobenius algebra over a field then $hd_R M=0$ or ∞ [2, Theorem 10]. Thus our theorem shows that for any $u(L)$-module M we have $hd_{u(L)}M=0$ or ∞. This is part of Theorem 5.1 of [1].

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