CONCERNING HAUSDORFF MATRICES AND ABSOLUTELY CONVERGENT SEQUENCES[1]

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Introduction. A complex number sequence $B = \{b_n\}$ will be called absolutely convergent [2, p. 72] if

$$\sum_p |b_p - b_{p+1}| < \infty.$$

If the function g is Riemann integrable [1, p. 192] on [0, 1], the generalized Hausdorff matrix, H^g, determined by g is defined by

$$H^g_{n,p} = \begin{cases} 0 & \text{if } p > n \\ \displaystyle\int_{[0,1]} \binom{n}{p} x^p(1-x)^{n-p} dg(x) & \text{if } p \leqq n \end{cases}$$

for $\left.{n \atop p}\right\} = 0, 1, 2, \cdots$.

This paper is concerned with the determination of those generalized Hausdorff matrices which sum all absolutely convergent sequences. Theorem 2 gives a sufficiency condition stated in terms of the behavior of the points of continuity of the graph of g and Theorems 3 and 4 provide examples which serve to further describe functions which generate such matrices. Theorem 5 gives a necessary and sufficient condition in terms of moment sequences. Theorem 1 gives conditions which are necessary and sufficient for a semi-infinite complex number matrix to sum all absolutely convergent sequences and although the theorem is known [3], a brief proof is included here for completeness since the author does not know of one in the literature.

THEOREM 1. *If A is an infinite complex triangular matrix, then the following two statements are equivalent*:

(i) *If B is an absolutely convergent complex number sequence, then $A \cdot B = T$ converges*; *and*

(ii)(a) *there is a number K such that if n, j is a nonnegative integer pair, then*

$$\left| \sum_{p=0}^{j} A_{n,p} \right| < K,$$

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(b) *the sequence*

$$\left\{\sum_{p=0}^{n} A_{n,p}\right\}_{n=0}^{\infty}$$

converges, and

(c) *for each nonnegative integer p, the sequence*

$$\{A_{n,p}\}_{n=p}^{\infty}$$

converges.

PROOF. Suppose (i) is true. Then (ii)(b) and (ii)(c) follow as consequences of the absolute convergence of 1, 1, 1, $\cdots$, and sequences of the form 0, 0, 0, $\cdots$, 0, 1, 0, $\cdots$.

In order to show that (i) implies (ii)(a), it is first established that (i) implies the existence of a number M such that

$$(1) \qquad |A_{n,p}| < M; \quad \left.\begin{matrix} n \\ p \end{matrix}\right\} = 0, 1, 2, \cdots.$$

Since (i) implies (ii)(c), there is a positive number sequence C such that for each nonnegative integer p, $|A_{n,p}| < c_p$; $n=0, 1, 2, \cdots$. If (1) is not true, then there is an increasing sequence N of integers and an increasing sequence P of integers such that $p_0=0$ and $|A_{n_i,p_i}| > 4^i \sum_{j=0}^{p_{i-1}} c_j$; $i=1, 2, 3, \cdots$. Let B be the absolutely convergent sequence

$$b_j = \begin{cases} \dfrac{1}{2^i} & \text{if } j = p_i, \\ 0 & \text{if } j \neq p_i, \end{cases} \qquad i = 1, 2, 3, \cdots.$$

The sequence $T=A\cdot B$ is unbounded since the sequence C is unbounded if (1) is not true;

$$|t_{n_i}| = \left|\sum_{p=0}^{n_i} A_{n_i,p} b_p\right| = \left|\sum_{j=1}^{i} \frac{A_{n_i,p_j}}{2^j}\right| > \left|\frac{A_{n_i,p_i}}{2^i}\right| - \sum_{j=0}^{p_{i-1}} \frac{c_j}{2^j}$$

$$> 2^i \sum_{j=0}^{p_{i-1}} c_j\left(1 - \frac{1}{2^j}\right) > 2^i \sum_{j=0}^{p_{i-1}} c_j.$$

This contradicts the assumption that (i) is true, hence (1) is true.

If (ii)(a) is not true, then for each $c>0$ there is a nonnegative integer pair n, j such that $\left|\sum_{p=0}^{j} A_{n,p}\right| > c$. Let M be the number given in (1); and for each nonnegative integer k, let n_k, j_k be an integer pair such that

$$(2)\qquad \left|\sum_{p=0}^{j_0} A_{n_0,p}\right| > 4M \quad\text{and}\quad \left|\sum_{p=0}^{j_k} A_{n_k,p}\right| > 4^{k+1}M2n_{k-1}; \qquad k \geqq 1.$$

Let B be the absolutely convergent sequence such that for each nonnegative integer p,

$$b_p = \begin{cases} 1 & \text{if } p \leqq j_0. \\ \dfrac{1}{2^{k+1}} & \text{if } n_k \leqq p \leqq j_{k+1}; \qquad k = 0, 1, 2, \cdots. \\ 0 & \text{if } j_k < p < n_k. \end{cases}$$

Consider the sequence $T = A \cdot B$. Using (1)

$$|t_{n_k}| \geqq \frac{1}{2^k}\left|\sum_{p=n_{k-1}}^{j_k} A_{n_k,p}\right| - Mn_{k-1}; \qquad k = 1, 2, 3, \cdots;$$

and from (1) and (2) it follows that

$$\left|\sum_{p=n_{k-1}}^{j_k} A_{n_k,p}\right| > 4^k M n_{k-1}.$$

Therefore,

$$|t_{n_k}| > 2^k M n_{k-1} - M n_{k-1} > M n_{k-1} 2^{k-1},$$

from which it follows that T is unbounded, contrary to (i) being true. Thus, (i) implies (ii).

Suppose (ii) is true and B is an absolutely convergent sequence. If C is a constant term sequence and D is the sequence such that for each nonnegative integer n, $d_n = b_n - c$, then $A \cdot B = A \cdot C + A \cdot D$. Therefore, we shall suppose that B has limit 0.

There is a number K' such that

$$K' > \begin{cases} |b_0| + \sum_{p=0}^{\infty} |b_p - b_{p+1}| \\ \left|\sum_{p=0}^{j} A_{n,p}\right|, \quad \left.\begin{matrix} n \\ j \end{matrix}\right\} = 0, 1, 2, \cdots, \\ |A_{n,p}|, \quad \left.\begin{matrix} n \\ p \end{matrix}\right\} = 0, 1, 2, \cdots, \text{ and} \\ |b_n|, \quad n = 0, 1, 2, \cdots. \end{cases}$$

If $c>0$, let N' be a positive integer such that if $n>N'$, then

$$\sum_{p=n}^{\infty} | b_p - b_{p+1} | < \frac{c}{32K'} \quad \text{and} \quad | b_n | < \frac{c}{16K'} \cdot \tag{3}$$

For each nonnegative integer $m \leqq N'$, let j_m be a positive integer such that if $n>j_m$ and k is a positive integer, then

$$| A_{n,m} - A_{n+k,m} | < \frac{c}{16N'K'2^m} \cdot \tag{4}$$

Let

$$N = \sum_{m=0}^{N'} j_m.$$

If $n>N$ and $k=1, 2, 3, \cdots$, then

$$| t_n - t_{n+k} | = \left| \sum_{p=0}^{n} A_{n,p} b_p - \sum_{p=0}^{n+k} A_{n+k,p} b_p \right|. \tag{5}$$

By use of the triangle inequality and summation by parts, it follows that the right side of (5) is equal to or less than

$$\begin{aligned} &\sum_{j=0}^{n-1} \left(| b_j - b_{j+1} | \sum_{p=0}^{j} | A_{n,p} - A_{n+k,p} | \right) + | b_n | \left| \sum_{p=0}^{n} A_{n,p} \right| \\ &\quad + | b_{n+k} | \left| \sum_{p=0}^{n+k} A_{n+k,p} \right| + \sum_{j=n}^{n+k-1} \left(| b_j - b_{j+1} | \left| \sum_{p=0}^{j} A_{n,p} \right| \right). \end{aligned} \tag{6}$$

It follows from (3), (4), and (6) that $|t_n - t_{n+k}| < c$. Therefore, (ii) implies (i).

DEFINITION 1. $R[a, b]$ is the function set to which g belongs if and only if g is Riemann integrable on $[a, b]$.

DEFINITION 2. If $n, p, p \leqq n$, is a nonnegative integer pair, then

$$f_{n,p}(x) = \binom{n}{p} x^p (1-x)^{n-p}.$$

If g is in $R[0, 1]$, then $\int_{[0,1]} f_{n,p} dg$ exists.

Proofs for the following lemmas are omitted.

LEMMA 1. *Suppose*

(a) *g is in $R[0, 1]$ and*

(b) *K is a number such that if t is in $[0, 1]$ and g is continuous at t, then $|g(t)| < K$;*

then

$$\left|\int_{[0,1]} g df_{n,p}\right| \leqq K V_{[0,1]} f_{n,p},$$

where $V_{[0,1]}f_{n,p}$ *denotes the variation of* $f_{n,p}$ *on* $[0, 1]$.

LEMMA 2. *If* t *is in the sect* $(0, 1]$ *and* p *is a nonnegative integer, then* $f_{n,p}(x) \to 0$ *as* $n \to \infty$ *uniformly on* $[t, 1]$.

LEMMA 3. *If* $n, j, j \leqq n$, *is a nonnegative integer pair, then*

$$\sum_{p=0}^{j} f_{n,p}(x) \text{ is nonincreasing on } [0, 1].$$

LEMMA 4. *If* p *is a nonnegative integer and* t *is a number in* $(0, 1]$, *then there is a number* K *such that if* x *is in* $[t, 1]$ *and* $n = p, p+1, p+2, \cdots$, *then*

$$|f'_{n,p}(x)| < K.$$

DEFINITION 3. If g is in $R[0, 1]$, M_g is the point set to which the point p with abscissa x belongs if and only if x is in $[0, 1]$ and g is continuous at x.

THEOREM 2. *If* g *is in* $R[0, 1]$ *and* M_g *has only one limit point on the Y-axis, then* H^g *satisfies the conditions of Theorem* 1.

PROOF. Integration by parts plus Lemma 3 shows that H^g satisfies (ii)(a). That H^g satisfies (ii)(b) follows from the fact that for each nonnegative integer n

$$\sum_{p=0}^{n} H^g_{n,p} = \int_{[0,1]} 1 dg.$$

In order to show that H^g satisfies (ii)(c), we suppose that p is a nonnegative integer and $c > 0$. Let L be a number such that if x is in $[0, 1]$, then

$$|g(x)| < L.^2 \tag{7}$$

Let $(0, a)$ be the limit point of M_g which lies on the Y-axis. There is a $t > 0$ such that if x is in $[0, t]$ and g is continuous at x, then

$$|g(x) - a| < \frac{c}{L+1}\cdot \tag{8}$$

There is a positive integer N such that if $n > N$, then

[2] From the definition of the R-S integral used here, it follows that if g is in $R[0, 1]$, then g is bounded.

(9) $\qquad f_{n,p}(x)$ is decreasing on $[t, 1]$ and $f_{n,p}(t) < \dfrac{c}{L+1}$.

Suppose $p=0$. Integration by parts shows that

$$\int_{[0,t]} f_{n,0}dg = f_{n,0}(t)g(t) - g(0) - af_{n,0}(t) + a - \int_{[0,t]} (g-a)df_{n,0} \tag{10}$$

It follows from (7) and (9) that

$$|af_{n,0}(t)| < c \quad \text{and} \quad |g(t)f_{n,0}(t)| < c. \tag{11}$$

Also, from Lemma 1 and (8) it follows that

$$\left|\int_{[0,t]} (g-a)df_{n,0}\right| < cV_{[0,1]}f_{n,0} = c. \tag{12}$$

Integration by parts plus (9) shows that

$$\left|\int_{[t,1]} f_{n,0}dg\right| < |g(t)f_{n,0}(t)| + LV_{[t,1]}f_{n,0} < 2c. \tag{13}$$

It follows, by use of (10), (11), (12), and (13), that

$$\left|\int_{[0,1]} f_{n,0}dg - [a-g(0)]\right| < 5c.$$

If $p>0$, an argument similar to that above shows that

$$\int_{[0,1]} f_{n,p}dg \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, H^g satisfies (ii)(c).

THEOREM 3. *There is in $R[0, 1]$ a function h such that M_h has two limit points on the Y-axis and H^h satisfies the conditions of Theorem* 1.

PROOF. Let T be a decreasing sequence which lies in $(0, 1)$ and has 0 as its limit. By Lemma 4, there is a number sequence K such that if p is a nonnegative integer and x is in $[t_p, 1]$, then for $j=0, 1, 2, \cdots, p$ and $n=j, j+1, j+2, \cdots$

$$|f'_{n,j}(x)| < k_p. \tag{14}$$

Let S be a sequence such that for $p=0, 1, 2, \cdots$

$$s_{p+1} < t_p < s_n < 1$$

and

$$(s_p - t_p)k_p < \frac{1}{2^p}. \tag{15}$$

Let

$$h(x) = \begin{cases} 0 & \text{if } s_0 \leqq x \leqq 1, \\ 0 & \text{if } s_{p+1} \leqq x \leqq t_p,\ p = 0, 1, 2, \cdots, \\ 1 & \text{if } t_p < x < s_p,\ p = 0, 1, 2, \cdots, \\ 0 & \text{if } x = 0. \end{cases}$$

h is in $R[0, 1]$ and it was established in Theorem 2 that this alone is sufficient for H^h to satisfy (ii)(a) and (ii)(b) of Theorem 1.

In order to show that H^h satisfies (ii)(c), we suppose that p is a nonnegative integer and that $c>0$. There is a positive integer $N>p$ such that if $n>N$, then

$$\sum_{j=n}^{\infty} \frac{1}{2^j} < \frac{c}{8}. \tag{16}$$

There is a positive integer N' such that if $n>N'$, then

$$f_{n,p} \text{ is decreasing on } [t_N, 1] \text{ and } \quad |f_{n,p}(t_N)| < \frac{c}{8}. \tag{17}$$

Let $N''=N+N'$.

Suppose $n>N''$.

$$\begin{aligned} &\left| \int_{[0,1]} f_{n,p}dh \right| \\ &= \left| \int_{(0,t_N]} f_{n,p}dh + f_{n,p}(1)h(1) - f_{n,p}(t_N)h(t_N) - \int_{[t_N,1]} hdf_{n,p} \right|. \end{aligned} \tag{18}$$

But, $h(1)=0$, $h(t_N)=0$ and it follows from (17) that

$$\left| \int_{[t_N,1]} hdf_{n,p} \right| < \frac{c}{8}.$$

Therefore, the right side of (18) is less than

$$\left| \int_{[0,t_N]} f_{n,p}dh \right| + \frac{c}{4}. \tag{19}$$

There is an integer $M>N$ such that

$$V_{[0,t_M]}f_{n,p} < \frac{c}{4}. \tag{20}$$

Since $h(0)=0$ and $h(t_M)=0$, integration by parts plus the triangle inequality shows that

$$\left|\int_{[0,t_N]} f_{n,p}dh\right| \leqq \left|\int_{[0,t_M]} hdf_{n,p}\right| + \left|\int_{[t_M,t_N]} f_{n,p}dh\right|. \tag{21}$$

It follows from (20) that

$$\left|\int_{[0,t_M]} hdf_{n,p}\right| < \frac{c}{4}. \tag{22}$$

Since h is a step function on $[t_M, t_N]$, the triangle inequality shows that

$$\left|\int_{[t_M,t_N]} f_{n,p}dh\right| \leqq \sum_{j=0}^{M-N} \left| f_{n,p}(t_{M-j}) - f_{n,p}(s_{M-j})\right|.$$

N is greater than p. Therefore, if t is in $[t_{M-j}, t_N]$, then $|f'_{n,p}(t)| < k_{M-j}$ so that

$$\left| f_{n,p}(t_{M-j}) - f_{n,p}(s_{M-j})\right| < k_{M-j}\left| s_{M-j} - t_{M-j}\right|.$$

From (15),

$$\sum_{j=0}^{M-N} k_{M-j}\left| s_{M-j} - t_{M-j}\right| < \sum_{j=0}^{M-N} \frac{1}{2^{M-j}},$$

which by (16) is less than $c/8$ so that

$$\left|\int_{[t_M,t_N]} f_{n,p}dh\right| < \frac{c}{8}. \tag{23}$$

It follows then, from (18), (19), (21), (22), and (23), that

$$\left|\int_{[0,1]} f_{n,p}dh\right| < c.$$

Therefore, H^h satisfies (ii)(c).

THEOREM 4. *There is a function g in $R[0, 1]$ such that M_g has two limit points on the Y-axis and H^g does not satisfy the conditions of Theorem* 1.

PROOF. It is sufficient to find a g in $R[0, 1]$ such that

$$\left\{\int_{[0,1]} f_{n,1}dg\right\}_{n=1}^{\infty}$$

diverges.

For each positive integer n and each x in $[0, 1]$,

$$f_{n,1}(x) \leqq f_{n,1}\left(\frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^{n-1} \to e^{-1}.$$

There exist positive integer sequences N and J such that

$$n_p < j_p < n_{p+1}, \qquad p = 1, 2, 3, \cdots,$$

$$f_{j_p,1}\left(\frac{1}{n_{p+1}}\right) < \frac{e^{-1}}{8},$$

and

$$f_{j_p,1}\left(\frac{1}{n_p}\right) < \frac{e^{-1}}{8} \cdot$$

It follows from these conditions that

$$\begin{aligned} &f_{j_p,1} \text{ is increasing on } \left[0, \frac{1}{n_{p+1}}\right] \\ &\text{and is decreasing on } \left[\frac{1}{n_p}, 1\right]. \end{aligned} \tag{24}$$

Let

$$g(x) = \begin{cases} 0 & \text{if } \dfrac{1}{j_0} \leqq x \leqq 1 \\ 1 & \text{if } \dfrac{1}{j_{2p+1}} < x < \dfrac{1}{j_{2p}}, \ p = 0, 1, 2, \cdots, \\ 0 & \text{if } \dfrac{1}{j_{2p}} \leqq x \leqq \dfrac{1}{j_{2p-1}}, \ p = 1, 2, 3, \cdots, \\ 0 & \text{if } x = 0. \end{cases}$$

Suppose p is a positive integer.

$$\int_{[0,1]} f_{j_p,1}dg = -\int_{[0,1/n_{p+1}]} gdf_{j_p,1} - \int_{[1/n_{p+1}\ 1/n_p]} gdf_{j_p,1}$$
$$- \int_{[1/n,1]} gdf_{j_p,1}.$$

From (24) it follows that

$$\left.\begin{array}{l} \left| \int_{[0,1/n_{p+1}]} g df_{j_p,1} \right| \\ \left| \int_{[1/n_p,1]} g df_{j_p,1} \right| \end{array} \right\} < e^{-1}/8.$$

Also,

$$\begin{aligned} & -\int_{[1/n_{p+1},1/n_p]} g df_{j_p,1} \\ (25) \qquad & = + f_{j_p,1}(1/n_{p+1}) g(1/n_{p+1}) - f_{j_p,1}(1/n_p) g(1/n_p) \pm f_{j_p,1}(1/j_p), \end{aligned}$$

with the sign of the last term negative if p is even and positive if p is odd. Therefore, if p is even the right side of (25) is less than $-e^{-1}+e^{-1}/4$ and if p is odd the right side of (25) is greater than $e^{-1}-e^{-1}/4$ so that $\int_{[0,1]} f_{j_p,1} dg$ is less than $-e^{-1}/2$ if p is even and is greater than $e^{-1}/2$ if p is odd. Therefore, the sequence

$$\left\{ \int_{[0,1]} f_{n,1} dg \right\}_{n=1}^{\infty}$$

contains a divergent subsequence which shows that H^g does not satisfy (ii)(c).

THEOREM 5. *If g is in $R[0, 1]$ and C is the moment sequence determined by g, then H^g satisfies the conditions of Theorem 1 if and only if for each nonnegative integer p the sequence*

$$\left\{ \binom{n}{p} \Delta^{n-p} c_p \right\}_{n=p}^{\infty}$$

converges.

Since the conclusion follows readily from the conditions of statement (ii) Theorem 1, the proof is omitted.

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