AN EXAMPLE FOR HOMOTOPY GROUPS WITH COEFFICIENTS

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In this note we present two spaces X and Y all of whose homotopy groups are isomorphic, but whose homotopy groups with coefficients are not isomorphic for a certain coefficient group.[1] This example depends on the fact that the universal coefficient sequence [**2**], which relates the ordinary homotopy groups to those with coefficients, does not split. The spaces X and Y will be 1-connected, CW-complexes and the coefficient group will be Z_m, the integers modulo m.

We adopt the following notation: $M(G, p)$ denotes a Moore complex of type (G, p) (i.e., a space with a single nonvanishing homology group G in dimension p) and $K(G, p)$ denotes an Eilenberg-MacLane complex of type (G, p) (i.e., a space with a single nonvanishing homotopy group G in dimension p). Recall that $\pi_r(G; A)$, the rth homotopy group of the space A with coefficients in the group G, is the group of homotopy classes of base point preserving maps from $M(G, r)$ into A. If Z is the group of integers, then $\pi_r(Z; A) = \pi_r(A)$, the rth homotopy group of A. Finally we recall the universal coefficient theorem [**2**] which asserts the exactness of the following sequence

$$(1) \qquad 0 \to \mathrm{Ext}(G, \pi_{r+1}(A)) \to \pi_r(G; A) \to \mathrm{Hom}(G, \pi_r(A)) \to 0.$$

Now let $\overline{X} = M(Z_n, r)$ and let d be the greatest common divisor of m and n. We shall always assume that d is even, that 8 does not divide mn, and that r is an integer >2. Under these conditions Barratt [**1**] has shown that $\pi_r(Z_m; \overline{X}) \approx Z_{2d}$. In addition, it is well known that $\pi_{r+1}(\overline{X}) \approx Z_n \otimes Z_2 = Z_2$. Now let X be the space obtained from $\overline{X}$ by attaching cells to kill all homotopy groups in dimensions $\geqq r+2$. Thus

$$(2) \qquad \begin{aligned} \pi_i(X) &= 0 \text{ for all } i \leqq r-1 \text{ and } \geqq r+2, \\ \pi_r(X) &= Z_n \text{ and } \pi_{r+1}(X) = Z_2. \end{aligned}$$

Furthermore, since $M(Z_m, r)$ is an $(r+1)$-dimensional CW-complex and the $(r+2)$-skeleton of X is $\overline{X}$, it follows from standard cellular approximation arguments that $\pi_r(Z_m; X) \approx \pi_r(Z_m; \overline{X})$. Hence we have

$$(3) \qquad \pi_r(Z_m; X) \approx Z_{2d}.$$

Next let Y be a product of Eilenberg-MacLane spaces, $Y = K(Z_n, r)$

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[1] We have been informed that such an example was known to Eckmann and Hilton.

$\times K(Z_2, r+1)$. Since $\pi_i(Y) \approx \pi_i(K(Z_n, r)) \oplus \pi_i(K(Z_2, r+1))$, by (2) we have that $\pi_i(X)$ and $\pi_i(Y)$ are isomorphic for all i. Now we consider $\pi_r(Z_m; Y)$. Clearly $\pi_r(Z_m; Y) \approx \pi_r(Z_m; K(Z_n, r)) \oplus \pi_r(Z_m; K(Z_2, r+1))$. By (1), $\pi_r(Z_m; K(Z_n, r)) \approx \mathrm{Hom}(Z_m, Z_n) = Z_d$. Also by (1),

$$\pi_r(Z_m; K(Z_2, r+1)) \approx \mathrm{Ext}(Z_m, Z_2) = Z_2.$$

Thus

$$\pi_r(Z_m; Y) \approx Z_d \oplus Z_2. \tag{4}$$

A comparison of (3) and (4) shows that $\pi_r(Z_m; X)$ is not isomorphic to $\pi_r(Z_m; Y)$. But we have already seen that $\pi_i(X) \approx \pi_i(Y)$ for all i. This completes the example.

We observe that the spaces X and Y may be distinguished by invariants other than the rth homotopy group with coefficients in Z_m. For instance, it is easily seen that $H_{r+1}(X) = 0$ and $H_{r+1}(Y) = Z_2$.

In closing we note that the spaces X and Y serve as an example for homotopy groups with coefficients as defined by Katuta [**3**]. This is so since, for a finite coefficient group, Katuta's groups are the same as the ones we consider, except for a dimensional shift of one unit.

References

1. M. G. Barratt, *Track groups*. II, Proc. London Math. Soc. (3) **5** (1955), 285-329.

2. B. Eckmann and P. J. Hilton, *Groupes d'homotopie et dualité. Coefficients*, C. R. Acad. Sci. Paris **246** (1958), 2991–2993.

3. Y. Katuta, *Homotopy groups with coefficients*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **7** (1960), 5–24.

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