AN EXTENSION OF GREENDLINGER'S RESULTS ON THE WORD PROBLEM

SEYMOUR LIPSCHUTZ

1. **Introduction.** In 1960 Greendlinger [3] solved the word problem for sixth-groups (see §2). In this paper we first solve the extended (generalized) word problem for certain subgroups of sixth-groups. We are then able (using results of Neumann [6]) to solve the word problem for generalized free products of sixth-groups with the above subgroups amalgamated.

The author conjectures that analogous results can be proven for classes of groups similar to sixth-groups—groups studied by Britton [2], Schiek [7] and Tartakovskii [8].

2. **Notations and definitions.** *Capital letters* denote words and *lower case letters* denote generators. We say that W is *fully reduced* if it does not contain more than half of a relator and it is freely reduced. We say that W is *cyclically reduced* if every cyclic transform of W is freely reduced, and that W is *cyclically fully reduced* if every cyclic transform of W is fully reduced.

We say that the words A_i satisfy the *one-sixth condition* if they have the following two properties: (i) the A_i are cyclically reduced, and (ii) if B_i and B_j are cyclic transforms of A_i and A_j, then less than one-sixth of the length of the shorter one cancels in the product $B_i^{\pm 1}B_j^{\pm 1}$, unless the product is unity.

We now have (cf. Lipschutz [5] or Greendlinger [3]) the

DEFINITION. A group G is a sixth-group if it is finitely presented in the form

$$G = \mathrm{gp}(a_1, \cdots, a_n; R_1(a_\lambda) = 1, \cdots, R_m(a_\lambda) = 1),$$

where the set of relators R_i satisfy the one-sixth condition.

We use the notation:

$l(W)$ for the legnth of W,

$A = B$ means A and B are the same element of G,

$A \approx B$ means A is freely equal to B,

$A \equiv B$ means A is identical to B,

$A \wedge B$ means A does not react with B, that is, nothing cancels in the product AB.

NOTE. There is no loss in generality if we assume that, in the pres-

Received by the editors December 7, 1962.

entation of a sixth-group, the cyclic transforms and inverses of relators are also included in the set of relators.

3. **Preliminary lemmas.** We state without proof a main result (cf. Greendlinger [3, p. 82, generalization (1)]) on sixth-groups:

LEMMA 1. *If W is freely reduced and $W=1$, then W contains more than $\frac{5}{6}$ of a relator or W contains disjointly two subwords, each containing more than $\frac{3}{6}$ of a relator.*

REMARK. The extended word problem of a given subgroup H of a group G is to decide whether or not an arbitrary given element of G is also in H. Usually G is given by generators and defining relations, and H is the subgroup generated by a given set of words in the generators of G. The extended word problem reduces to the word problem when $H=1$, hence is unsolvable in general (cf. Boone [1, §35]).

We easily prove the next lemma using results of Neumann [6]:

LEMMA 2. *Let G be the free product of the groups*

$$G_i = \mathrm{gp}(a_{i1}, \cdots, a_{in}, b_{i1}, \cdots, b_{in_i}; R_{i1} = \cdots = R_{im_i} = 1) \qquad (i = 1, 2, \cdots)$$

with the subgroups $H_i=\mathrm{gp}(a_{i1}, \cdots, a_{in})$ amalgamated in the obvious way, that is, the presentation of G consists of the union of the generators and the defining relations of the G_i and the added defining relations

$$a_{1\lambda} = a_{2\lambda} = \cdots \qquad (\lambda = 1, 2, \cdots, n).$$

Then G admits a solution to its word problem if the following are known:

(a) *The word problem has been solved for the groups G_i.*

(b) *The extended word problem has been solved for the subgroups H_i in G_i.*

PROOF. Let W be in G. Then we can write

$$W \equiv W_1 \cdots W_n,$$

where each "factor" W_i is in some group G_j and no successive pair of factors W_i, W_{i+1} belong to the same group. If the factors W_i are not contained in the amalgamated subgroups then the (generalized free product) length of W is $n>0$ (cf. Neumann [6]) and $W\neq 1$. If a factor, say W_j, is in an H_i then we can find a word $V(a_{i\lambda})$ such that $W_i=V$. Then

$$W = W_1 \cdots W_{j-1}V(a_{k\lambda})W_{j+1} \cdots W_n$$

is a product of less then n factors. By continuing this process we can find the "length" of W and, in particular, determine if $W=1$.

The next two lemmas are about sixth-groups G. As the proofs are relatively long and combinatorial, we give them in §5 and §6.

LEMMA 3. *Let* $l(W)=n$ *and* $W\neq 1$. *If* U *is fully reduced and* $U=W$, *then* $l(U)\leqq rn$, *where* r *is the length of the largest relator in the sixth-group* G.

LEMMA 4. *Let* W *be cyclically fully reduced and of infinite order. If* W^2 *is also cyclically fully reduced then* W^n *is fully reduced for all* n. *If* W^2 *is not cyclically fully reduced then there exists a relator*

$$R \equiv W_1W_2W_1T^{-1},$$

where W_1W_2 *is a cyclic transform of* W, $l(T)<\frac{1}{2}l(R)$ *and* $(TW_2)^n$ *is fully reduced for all* n.

4. Main results.

THEOREM 1. *Let*

$$G = \mathrm{gp}(a_1, \cdots, a_n, b_1, \cdots, b_m; R_1 = \cdots = R_p = 1)$$

be a sixth-group, where every freely reduced word $W=W(a_\lambda)$ *is fully reduced. Then* $H=\mathrm{gp}(a_1, \cdots, a_n)$ *is a free subgroup of* G *with the* a_i *as free generators and one can solve the extended word problem with respect to* H.

PROOF. By Greendlinger's Lemma 1, every word $W(a_\lambda)\neq 1$ so H is free. Let V be in G. If V is in H, that is, if $V=W(a_1, \cdots, a_n)$ then, by Lemma 3, we know the maximum length of W. Since there are only a finite number of words $W(a_\lambda)$ of any given length and since the word problem has been solved for sixth-groups, the theorem is true.

THEOREM 2. *Let* W *be any element in a sixth-group* G. *Then one can solve the extended word problem with respect to the subgroup* $H=\mathrm{gp}(W)$.

PROOF. By a theorem of Greendlinger [**3**, p. 668], we can find the order of W. If the order of W is finite, say n, then any word V in G is also in H iff there exists an m, $1\leqq m\leqq n$, such that $V=W^m$. Since the word problem is solvable in G, this case is decidable.

Suppose W has infinite order. Since V is in $\mathrm{gp}(W)$ iff AVA^{-1} is in $\mathrm{gp}(AWA^{-1})$, we can reduce our problem, by taking an appropriate conjugate of V and W, to the case where W is cyclically fully reduced and has the properties of Lemma 4.

If W^n is fully reduced for all n, then our theorem, as in Theorem 1, follows from Lemma 3. If W^n is not fully reduced then, by Lemma 4,

$$W^n \equiv (W_1W_2W_1W_2)^mW^\epsilon = (TW_2)^mW^\epsilon,$$

where $\epsilon=1$ or $\epsilon=0$. Hence V is in $\mathrm{gp}(W)$ iff V or VW^{-1} is in $\mathrm{gp}(TW_2)$. But $(TW_2)^m$ is fully reduced for all m. So the theorem is true for this case also.

The next two theorems follow directly from the previous theorems and Lemma 2.

THEOREM 3. *In the notation of Lemma 2 suppose that each G_i is a sixth-group and any freely reduced word $W(a_{i\lambda})$ is fully reduced. Then the generalized free product G of the G_i amalgamating the subgroups H_i has a solvable word problem.*

THEOREM 4. *Let $G_1, G_2, \cdots$ be sixth-groups. Let H_i be a cyclic subgroup of G_i generated by W_i $(i=1, 2, \cdots)$. If the orders of the W_i are equal then the generalized free product G of the G_i amalgamating the subgroups H_i has a solvable word problem.*

THEOREM 5. *Let $\mathcal{G}_1$ consist of groups G which admit solutions to their word problems and to their extended word problems with respect to the infinite cyclic group generated by any element W in G of infinite order. Let $\mathcal{G}_k$, $k>1$, consist of groups G which are the generalized free products of groups in $\mathcal{G}_{k-1}$ with an infinite cyclic group amalgamated. Then any group G in $\mathcal{G}_k$ admits a solution to its word problem, and the extended word problem with respect to any infinite cyclic subgroup is solvable.*

PROOF. In view of Lemma 2, we need only solve the extended word problem for G in $\mathcal{G}_k$, $k>1$, with respect to the infinite cyclic group generated by, say,

$$W \equiv W_1W_2 \cdots W_n,$$

where the W_i are not in the amalgamated subgroups and W_i, W_{i+1} do not belong to the same group in $\mathcal{G}_{k-1}$, that is, the length of W is $n>0$. By taking an appropriate transform of W, we can further assume, without loss in generality, that W is cyclically reduced, that is, W_1 and W_n are also in different groups. Then the length of W^m is precisely mn. Let V be in G. By the process of Lemma 2, we can determine the length of V. Since the word problem has been solved in G by Lemma 2 and the inductive hypothesis, we can decide if $V=W^m$ for some m, that is, if V is in $\mathrm{gp}(W)$.

COROLLARY 1. *If $\mathcal{G}_1$ is the class of sixth-groups and $\mathcal{G}_k$, $k>1$, is defined as in Theorem 5, then group G in $\mathcal{G}_k$ admits a solution to its word problem.*

5. Proof of Lemma 3. Let W be a word of minimum length for which the lemma is not true, that is, $W=V^{-1}$, where $l(V)>nr$ and

V is fully reduced. So

$$WV = 1. \tag{1}$$

The minimality of W guarantees that W is also fully reduced and that (1) is freely reduced. Thus (1) must satisfy Lemma 1; in particular, since W and V are fully reduced, WV must contain $>\frac{5}{6}$ of a relator R. Say $W \equiv AB$, $V \equiv CD$, $S \equiv BC$, where $R \equiv SE^{-1} \equiv BCE^{-1}$ and $l(S) > \frac{5}{6}l(R)$.

Now, substituting in (1), we have

$$WV \equiv ABCD \equiv ASD = AED = 1.$$

Notice that:

$l(D) > r(n-1)$,

$D \neq 1$ since $l(D) > 0$ and D is fully reduced,

$l(C) \leqq \frac{3}{6}l(R)$ since V is fully reduced,

$l(B) > \frac{2}{6}l(R)$ since $l(S) > \frac{5}{6}l(R)$,

$l(E) < \frac{1}{6}l(R)$ since $l(S) > \frac{5}{6}l(R)$.

Thus $l(E) < l(B)$, which implies $l(AE) < l(AB) = l(W)$. But $AE = D^{-1}$ also violates Lemma 3. This contradicts the minimality of W, so our lemma is true.

6. **Proof of Lemma 4.** The following remark is easily proven for sixth-groups. Suppose there is a relator $R' \equiv A^nB$, where $n > 1$. Then either

$$l(A) \leqq l(A^{n-1}) < \frac{1}{6}l(R')$$

or there exists a word C such that $A \equiv C^s$, $B \equiv C^t$ and, therefore, $R' \equiv C^m$. For the cyclic transform $A^{-1}B^{-1}A^{1-n}$ of R'^{-1} absorbs A^{n-1} from R'.

Let W^n contain more than half of a relator, say S, where $R \equiv ST^{-1}$ and $l(S) > \frac{1}{2}l(R)$. If we show that S must be contained in V^2, where V is a cyclic transform of W, then this proves the first part of the lemma. Suppose S is not contained in any V^2. Then $S \equiv V^rA$, where $r > 1$ and $V \equiv AB$. Consequently $R \equiv V^rAT^{-1}$. By the previous remark, either

$$l(S) = l(V^rA) = l(V^{r-1}) + l(V) + l(A) \leqq \frac{3}{6}l(R)$$

or, for some C, $V \equiv C^s$ and $R \equiv C^m$. In the first case we contradict the fact that S is more than half of R and in the second case we contradict the fact that W is of infinite order. Thus S must be contained in V^2.

We can now assume without loss in generality that $S \equiv W_1W_2W_1$, $V \equiv W_1W_2$ and $R \equiv W_1W_2W_1T^{-1}$. Since W is cyclically fully reduced,

$l(V) \leqq \frac{1}{2}l(R)$ which implies that $T \not\equiv 1$. Also, $W_2 \not\equiv 1$ else $R \not\equiv V^2T^{-1}$ and, by the remark about sixth-groups, W will be of finite order. We can further assume, by maximizing the possible length of S, that $W_2 \wedge T$ and $T \wedge W_2$. Note also that $W_1 \wedge W_2$ and $W_2 \wedge W_1$, since W is fully reduced.

Next we note that $W_2^{-1}W_1^{-1}TW_1^{-1}$, which absorbs W_1 from R, cannot be the inverse of R since $T \wedge W_2$; hence $l(W_1) < \frac{1}{6}l(R)$. This last inequality can be used to show, by simple arithmetic arguments, that

$$l(W_2),\ l(T) > \tfrac{1}{6}l(R).$$

We are now ready to prove that $(TW_2)^n$ is fully reduced for all n, which will prove our lemma.

Suppose $(TW_2)^n$ contains Q, where Q is more than half of a relator, say $R^* \equiv QP$ and $l(Q) > \frac{1}{2}l(R^*)$. There are four possibilities:

Case I. Q contains T, say $Q \equiv MTN$.

Then $R^{*\prime} \equiv TNPM$ is the inverse of $R \equiv W_1W_2W_1T^{-1}$ since more than one-sixth of R, i.e. T, is absorbed in the product of R with $R^{*\prime}$. But this contradicts:

$W_1 \wedge W_2$ if N is not empty,

$W_2 \wedge W_1$ if M is not empty,

$l(T) \leqq \frac{1}{2}l(R)$ if M and N are empty.

Thus Q does not contain T.

Case II. Q contains W_2.

Since W_2 is more than one-sixth of R, this case is also impossible as in Case I.

Case III. TW_2 contains Q, say $Q \equiv T^\alpha W_2^\alpha$ where $T \equiv T^\beta T^\alpha$ and $W_2 \equiv W_2^\alpha W_2^\beta$.

Note that $l(T^\alpha)$ or $l(W_2^\alpha) > \frac{1}{6}l(R^*)$. Then a cyclic transform of

$$R \equiv W_1W_2^\alpha W_2^\beta W_1(T^\alpha)^{-1}(T^\beta)^{-1}$$

will absorb more than $\frac{1}{6}$ of $R^* \equiv T^\alpha W_2^\alpha P$ or R^{*-1}. That these relators cannot be inverses follows either from the fact that $W_1 \wedge W_2$ or from the fact that $W_1 \wedge T^{-1}$. Accordingly, this case is impossible.

Case IV. W_2T contains Q.

Impossible as in Case III.

Thus we have proven our lemma.

Bibliography

1. W. Boone, *The word problem*, Ann. of Math. (2) **70** (1959), 207–265.

2. J. L. Britton, *Solution of the word problem for certain types of groups*, Proc. Glasgow Math. Assoc. **3** (1956), 45–54.

3. M. Greendlinger, *Dehn's algorithm for the word problem*, Comm. Pure Appl. Math. **13** (1960), 67–83.

4. ———, *On Dehn's algorithm for the conjugacy and word problems, with applications*, Comm. Pure Appl. Math. **13** (1960), 641–677.

5. S. Lipschutz, *On powers of elements in S-groups*, Proc. Amer. Math. Soc. **13** (1962), 181–186.

6. B. H. Neumann, *An essay on free products of groups with amalgamations*, Philos. Trans. Roy. Soc. London Ser. A **246** (1954), 503–554.

7. H. Schiek, *Ähnlichkeitsanalyse von Gruppenrelationen*, Acta Math. **96** (1956), 156–252.

8. V. A. Tartakovskiĭ, *Solution of the word problem for groups with a k-reducible basis for* $k>6$, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 483–494.

Polytechnic Institute of Brooklyn