GENERALIZED LAURENT SERIES FOR SINGULAR SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS[1]

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Introduction. Let $\bar{L}$ be a linear elliptic differential operator with analytic coefficients in a region R of E_n. Let L be the adjoint of $\bar{L}$. This paper extends the previous work of F. John[2] on representation of a solution u of $\bar{L}[u]=0$, where u has a singularity of *finite* order. A representation is developed here for a solution v of $\bar{L}[v]=0$, where v has an *isolated essential* singularity. This representation is a generalization of the Laurent series. Here the summation over the nth powers is replaced by summation over the nth derivatives of a fundamental solution $K(x, z)$, of the operator L. The representation in general is not unique.

Uniqueness of a suitably normalized representation is proved for the case in which $\bar{L}$ is homogeneous with constant coefficients. This gives rise to a theorem which for the three-dimensional Laplace operator reduces to the Maxwell-Sylvester theorem.[3]

The general case. Let $v(x)$ be a solution of $\bar{L}[v(x)]=0$, which has an isolated essential singularity at $x=y$, an interior point of the real region R, but is otherwise regular in the deleted region R. Let $\bar{L}$ be of order m.

Let $\mathfrak{D}\subset R$ be an open annular domain about the point $x=y$. Let S_1 be the sphere bounding the outer ball B_1 of $\mathfrak{D}$ and let S_2 be the sphere bounding the inner ball B_2 of $\mathfrak{D}$. Both B_1 and B_2 have the point $x=y$ as center and $B_1\neq B_2$. We further assume that B_1 is so small that $K(x, z)$, a fundamental solution of L, is analytic for $x\neq z$ in B_1.

THEOREM I. *For $z\in\mathfrak{D}$, $v(z)$ permits the following representation:*

$$v(z) = \omega(z) - \sum_{\nu=0}^{\infty}\sum_{|i|=\nu} A_i D^i K(x, z)\Big|_{x=y},$$

where $\omega(z)$ is analytic for $z\in B_1$, A_i are constants depending on S_2, $|i|=i_1+i_2+\cdots+i_n$ and

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[1] I am indebted to Professor Fritz John for suggesting this problem.

[2] See [1].

[3] See [2, pp. 514–521].

$$D^i = \frac{\partial^{|i|}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}.$$

PROOF. We know that there exists a ball $B_3 \subset B_2$ with center at $x=y$ such that $K(x, z)$ has a Taylor series expansion with respect to x, about $x=y$, for $x \in B_3$, $z \in \mathfrak{D}$. Let S_3 be the sphere bounding B_3 and let $\bar{R}$ be the annular region defined by S_1 and S_3. $\bar{R}$ has the boundary $\beta = S_1 \cup S_3$. Applying Green's identity to the operator $\bar{L}$ over $\bar{R}$ we get for $z \in \mathfrak{D}$:

$$\begin{aligned} v(z) &= \int_\beta M[K(x, z), v(x)] dS_x \\ &= \int_{x \in S_1} M[K(x, z), v(x)] dS_x - \int_{x \in S_3} M[K(x, z), v(x)] dS_x, \end{aligned} \tag{1}$$

where M is a bilinear operator, and dS_x denotes integration over the surface of the boundary.

$$\omega(z) = \int_{x \in S_1} M[K(x, z), v(x)] dS_x$$

is clearly analytic for $z \in \mathfrak{D}$, since $x \in S_1$ and therefore $x \neq z$.

For $z \in \mathfrak{D}$, $x \in S_3$, $K(x, z)$ has the Taylor series expansion about $x=y$:

$$K(x, z) = \sum_{\nu=0}^{\infty} \sum_{|i|=\nu} \frac{1}{i!} (x-y)^i D_y^i K(y, z),$$

where $D_y^i K(y, z) = D^i K(x, z)|_{x=y}$; $(x-y)^i = (x_1 - y_1)^{i_1} \cdots (x_n - y_n)^{i_n}$; and $i! = (i_1!) \cdots (i_n!)$. Equation (1) then becomes:

$$v(z) = \omega(z) - \int_{x \in S_3} M\left[\sum_{\nu=0}^{\infty} \sum_{|i|=\nu} \frac{1}{i!} (x-y)^i D_y^i K(y, z), v(x) \right] dS_x.$$

Since M is a linear operator and the series is uniformly convergent:

$$v(z) = \omega(z) - \sum_{\nu=0}^{\infty} \sum_{|i|=\nu} \frac{1}{i!} D_y^i K(y, z) \int_{x \in S_3} M[(x-y)^i, v(x)] dS_x.$$

Let

$$A_i = \frac{1}{i!} \int_{x \in S_3} M[(x-y)^i, v(x)] dS_x.$$

A_i is a constant. We then have as a final expression:

$$v(z) = \omega(z) - \sum_{\nu=0}^{\infty} \sum_{|i|=\nu} A_i D_y^i K(y, z). \tag{2}$$

It would be desirable to normalize equation (2), i.e., to obtain a form of equation (2) in which the A_i are *independent* of the spheres S_2 and S_3. Equation (2) would then be unique and hold for any $z \neq y$, $z \in B_1$; because for any such z, we can find a B_2 and a B_3 with radius so small that the required Taylor series expansion for $K(x, z)$ will exist.

We shall succeed in giving such a normalization of equation (2) only for the special case of $\bar{L}$ homogeneous with constant coefficients, i.e.,

$$\bar{L} = \sum_{|i|=m} b_i D^i, \tag{3}$$

where the b_i are real constants. Let the coefficient of

$$D^I = \frac{\partial^m}{\partial x_1^m}$$

in $\bar{L}$ be b_I. Since $\bar{L}$ is elliptic b_I is not zero. We know that there exists a fundamental solution of $\bar{L}$ of the form $K(y-x)$,[4] we shall use this fundamental solution in Theorem II.

THEOREM II.[5] *Let $v(x)$ be a solution of $\bar{L}[v(x)] = \sum_{|i|=m} b_i D^i v(x) = 0$ for $x \in B_1$, $x \neq y$. Let v have an isolated essential singularity at $x = y$. Then there exists a representation of $v(z)$ of the form:*

$$v(z) = \omega(z) - \sum_{\nu=0}^{\infty} \sum_{|s|=\nu} A_s D_y^s K(y - z), \qquad \textit{for } z \neq y,\ z \in B_1,$$

where $A_s = 0$, whenever D_y^s contains $D_y^I = \partial^m/\partial y_1^m$ as factor (i.e., whenever $s_1 \geqq m$); and this representation is unique.

PROOF. For $z \neq y$, $z \in \mathfrak{D}$, using equation (2) we can write:

$$v(z) = \omega_1(z) - \sum_{\nu=0}^{\infty} \sum_{|k|=\nu} A_k D_y^k [K(y - z)]. \tag{4}$$

We will prove that each operator term $\sum_{|k|=\nu} A_k D_y^k$ in the above series, with $\nu \geqq m$ can be written in the following form when it is applied to any solution u of $\bar{L}[u] = 0$.

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[4] See [1, pp. 298–303].

[5] This is a generalization of the Maxwell-Sylvester theorem for the case of the three-dimensional Laplacian. See [2, pp. 514–521].

$$\sum_{|k|=\nu} A_k D^k u = \sum_{|s|=\nu;\, s_1<m} J_s D^s u, \tag{5}$$

where J_s are constants. We will also prove that equation (5) is unique. From equation (3) we have:

$$D^I u = \frac{1}{b_I}\left[\bar{L} - \sum_{|i|=m;\, i\neq I} b_i D^i\right] u = - \sum_{|i|=m;\, i\neq I} \frac{b_i D^i u}{b_I}. \tag{6}$$

The operator term $\sum_{|k|=\nu} A_k D^k$ can be written as a polynomial in powers of $\partial/\partial x_1$, i.e.,

$$\sum_{|k|=\nu} A_k D^k = \sum_{j=0;\, j+|\mu|=\nu;\, \mu_1=0}^{p} \sum_{\mu} G_{\mu+j} D^{\mu} \frac{\partial^j}{\partial x_1^j}, \tag{7}$$

where $G_{\mu+j}$ are constants,[6] p a non-negative integer which is the highest order of $\partial/\partial x_1$ in $\sum_{|k|=\nu} A_k D^k$. For solutions u of $\bar{L}[u]=0$ and for $p \geqq m$, using equation (6), we can reduce the pth order polynomial operator in $\partial/\partial x_1$ in equation (7) to a $(p-1)$th order polynomial in $\partial/\partial x_1$. It is then seen that after $p-m+1$ steps equation (7), when applied to solutions u of $\bar{L}[u]=0$ reduces to:

$$\sum_{|k|=\nu} A_k D^k u = \sum_{j=0;\, j+|\mu|=\nu;\, \mu_1=0}^{m-1} \sum_{\mu} H_{\mu+j} D^{\mu} \frac{\partial^j}{\partial x_1^j} u = \sum_{|s|=\nu;\, s_1<m} J_s D^s u, \tag{8}$$

where $H_{\mu+j}$, J_s are constants.

We now prove uniqueness. Suppose we had the two identities, for all u with $\bar{L}[u]=0$:

$$\sum_{|k|=\nu} A_k D^k u = \sum_{|s|=\nu;\, s_1<m} J_s D^s u \tag{5}$$

and

$$\sum_{|k|=\nu} A_k D^k u = \sum_{|s|=\nu;\, s_1<m} J_s' D^s u; \tag{9}$$

then

$$\hat{L}[u] = \sum_{|s|=\nu;\, s_1<m} (J_s - J_s') D^s u = 0. \tag{10}$$

Equation (10) holds for all solutions u of $\bar{L}[u]=0$. Consider solutions u of the form:

$$u(x) = f(x_1) e^{\sigma_2 x_2 + \cdots + \sigma_n x_n}, \tag{11}$$

where the σ_i are constants. For a given set of σ_i, $\bar{L}[u(x)]=0$ is an mth order differential equation in the variable x_1, and there are m

[6] The index $\mu+j = (\mu_1+j, \mu_2, \cdots, \mu_n)$.

independent solutions, $f(x_1)$. If we apply $\hat{L}$ to $u(x)$ as defined in equation (11), then for a given set of σ_i we get either at most $m-1$ independent solutions for $f(x_1)$ or that the coefficients of all derivatives of $f(x_1)$ in $\hat{L}$ vanish.

Since equation (10) is to hold for all solutions u of $\bar{L}[u]=0$, it is impossible for $\hat{L}[u]=0$ to give only $m-1$ solutions for $f(x_1)$. On the other hand, if the coefficients of all orders of $\partial/\partial x_1$ vanish in equation (10) for all sets of σ_i, they vanish identically. We therefore conclude that $J_s=J'_s$, i.e., equation (5) is unique.

We now apply identity (5) to $u=K(y-z)$, which satisfies $\bar{L}_y[K(y-z)]=0$ for $z\neq y$, since L is self-adjoint. Substituting in equation (4) we get the normalized representation:

$$(12)\quad v(z) = \omega_1(z) - \sum_{\nu=0}^{\infty} \sum_{|s|=\nu;\, s_1<m} J_s D_y^s K(y-z) \qquad \text{for } z \neq y,\ z \in \mathfrak{D},$$

where we have defined $A_s=J_s$ for $|s|<m$.

We now prove that equation (12) is a unique representation of $v(z)$.[7] In particular it is independent of the construction leading to the coefficients A_i. For suppose we had another representation of $v(z)$:

$$(13)\quad v(z) = \omega_2(z) - \sum_{\nu=0}^{\infty} \sum_{|s|=\nu;\, s_1<m} J'_s D_y^s K(y-z), \qquad \text{for } z \neq y,\ z \in \mathfrak{D}',$$

where $\mathfrak{D}'$ is some annular region about the point $x=y$ such that $\mathfrak{D}\cap\mathfrak{D}'=\mathfrak{D}''$ is a nonempty set. Let the outer ball of $\mathfrak{D}''$ be B''_0 with radius r_0, and the inner ball be B''_I with radius r_I.

Subtracting equation (13) from equation (12) we get for $z\neq y$, $z\in\mathfrak{D}''$:

$$(14)\quad \phi(z) = \omega_1(z) - \omega_2(z) = \sum_{\nu=0}^{\infty} \sum_{|s|=\nu;\, s_1<m} (J_s - J'_s) D_y^s K(y-z).$$

We want to prove that $J_s=J'_s$ and $\omega_1(z)=\omega_2(z)$. We know the form of the fundamental solution $K(y-z)$.[8]

$$K(y-z) = \begin{cases} \rho^{m-n} A\left(\dfrac{z-y}{\rho}\right) & \text{for odd } n, \\[2ex] \rho^{m-n}\left[B\left(\dfrac{z-y}{\rho}\right)\log\rho + C\left(\dfrac{z-y}{\rho}\right)\right] & \text{for even } n, \end{cases}$$

[7] Unique within the choice of x_1.

[8] See [1, pp. 298–303].

where $\rho=|z-y|$, and A, B, C, $K(y-z)$ are analytic for all real y, z with $y\neq z$. For n even $\rho^{m-n}B((z-y)/\rho)$ is a polynomial in $z-y$. Then $D_y^s K(y-z)$ has the form:

$$D_y^s K(y-z) = \begin{cases} \rho^{m-n-|s|}E_s\left(\dfrac{z-y}{\rho}\right) & \text{for odd } n, \\ \rho^{m-n-|s|}\left[H_s\left(\dfrac{z-y}{\rho}\right)\log\rho + G_s\left(\dfrac{z-y}{\rho}\right)\right] & \text{for even } n, \end{cases}$$

E_s, H_s, G_s are regular for all real $z\neq y$. We can then write equation (14) in the form

$$\begin{aligned} \phi(z) &= \sum_{\nu=0}^{\infty} \sum_{|s|=\nu;\, s_1<m} (J_s - J_s')D_y^s K(y-z) \\ &= \sum_{\nu=0}^{\infty} \rho^{m-n-\nu}N_\nu\left(\frac{z-y}{\rho}\right) - Q\left(\rho, \frac{z-v}{\rho}\right)\log\rho. \end{aligned} \tag{15}$$

Let

$$\eta = \frac{z-y}{\rho};$$

then

$$\phi(y+\rho\eta) = \sum_{\nu=0}^{\infty} \rho^{m-n-\nu}N_\nu(\eta) - Q(\rho,\eta)\log\rho, \tag{16}$$

where $Q(\rho,\eta)$ is a polynomial in $z-y$ and $Q\equiv 0$ for odd n.

Consider an analytic continuation of equation (16) from the real ρ to the complex $\zeta=\rho+i\xi$. For any fixed η, both $\phi(y+\zeta\eta)$ and $\sum_{\nu=0}^{\infty} \zeta^{m-n-\nu}N_\nu(\eta)$ are univalued functions, analytic in ζ, and $Q(\zeta,\eta)\log\zeta$ is a multivalued function in $\mathfrak{D}''$. Since this is true for every fixed η, we conclude that for both odd and even n,

$$\phi(y+\rho\eta) = \sum_{\nu=0}^{\infty} \rho^{m-n-\nu}N_\nu(\eta). \tag{17}$$

For any fixed η, the right-hand side of equation (17) converges not only for $r_0>\rho>r_I$ but for all $\infty>\rho\geqq r_I$. This is so because the series part of equation (17) becomes a power series in negative powers of ρ, except possibly for a finite number of positive powers in the case of $m-n>0$. $\phi(y+\rho\eta)$ is then analytic for all $\infty>\rho\geqq 0$. From the analyticity of $\phi(y+\rho\eta)$ in $\infty>\rho\geqq 0$, we conclude, that the portion of the series in equation (17) containing negative powers of ρ must vanish, i.e.,

$$\sum_{\nu=0;\, m-n-\nu<0}^{\infty} \rho^{m-n-\nu} N_\nu(\eta) = 0. \tag{18}$$

For $m-n<0$ this means that $\phi(z)=0$. When $m-n\geqq 0$ we get in addition:

$$\sum_{\nu=0;\, m-n\geq 0}^{m-n} \rho^{m-n-\nu} N_\nu(\eta) = \phi(y+\rho\eta). \tag{19}$$

We will take these two cases separately. First for equation (19), using equation (15), we get:

$$\phi(z) = \sum_{\nu=0;\, m-n\geqq 0}^{m-n} \sum_{|s|=\nu} (J_s - J_s')D_y^s K(y-z) = \tilde{L}_y[K(y-z)] \tag{20}$$

Since $|s| \leqq m-n$, from the form of $D_y^s K(y-z)$ we see that equation (20) holds even at $y=z$. Let $\overline{B}$ be a ball containing y as an interior point. Let $\psi(\eta)$ be any regular function which vanishes in a neighborhood of $\bar{\beta}$, the boundary of $\overline{B}$. By Green's identity

$$\psi(\eta) = \int_{\overline{B}} \overline{L}[\psi(z)]K(\eta - z)dz;$$

then

$$\tilde{L}[\psi(\eta)]\Big|_{\eta=y} = \tilde{L}_y[\psi(y)] = \int_{\overline{B}} \overline{L}[\psi(z)]\phi(z)dz = \int_{\overline{B}} \psi(z)\overline{L}[\phi(z)]dz. \tag{21}$$

Since L is self-adjoint and ϕ is a solution of $\overline{L}[\phi]=0$, equation (21) implies that every regular function ψ, which vanishes in a neighborhood of $\bar{\beta}$ has the property that at $x=y$, $\tilde{L}[\psi]=0$. We then conclude that $\tilde{L}\equiv 0$ and therefore from equation (20) $\phi\equiv 0$, i.e., $\omega_1(z)=\omega_2(z)$. Also for $m-n\geqq 0$, and $m-n\geqq |s| \geqq 0$, we have $J_s = J_s'$.

We now return to equation (18). Since for every fixed η equation (18) is a power series its terms vanish separately. From equation (15) this means that:

$$\sum_{|s|=\nu;\, m-n-\nu<0} (J_s - J_s')D_y^s K(y-z) = 0. \tag{22}$$

This could be written:

$$\begin{aligned} \sum_{|s|=\nu;\, m-n-\nu<0} (J_s - J_s')D_y^s K(y-z) &= \pm \sum_{|s|=\nu;\, m-n-\nu<0} (J_s - J_s')D_z^s K(y-z) \\ &= P_z[K(y-z)] = 0. \end{aligned} \tag{23}$$

P_s is a linear homogeneous operator with constant coefficients, and equation (23) holds for all z such that $r_I \leqq |y-z|$. But $K(y-z)$ is analytic for all real z with $z \neq y$. Then equation (21) holds for all real z with $z \neq y$.

Let G be a simply connected open finite region. Let β_G be the boundary of G. Then a regular solution f, of $\bar{L}[f]=0$ has a representation in G:

$$f(z) = \int_{\beta_G} M[K(\xi - z), f(\xi)]dS_\xi$$

and so

$$P_s[f(z)] = \int_{\beta_G} M[P_s[K(\xi - z)], f(\xi)]dS_\xi = 0 \text{ in } G.$$

Then every regular solution f of $\bar{L}[f]=0$ in G is also a solution of $P_s[f(z)]=0$ in G. But P_s has the property that all D_z^s in P_s have $s_1 \leqq m-1$. It has been proved previously that not every $f(z)$ which solves $\bar{L}_z[f(z)]=0$ can solve $P_z[f(z)]=0$ unless $P_s \equiv 0$. We then conclude that $J_s = J_s'$ for all $|s| \geqq 0$, and that equation (12) is a unique representation of equation (2). Furthermore, this representation holds right up to the singularity, since in our original construction we may now take $r_2 > 0$, the radius of S_2 as small as we wish.

Bibliography

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