2. G. Higman, *On finite groups of exponent five*, Proc. Cambridge Philos. Soc. **52** (1956), 381–390.

3. A. I. Kostrikin, *On Burnside's problem*, Dokl. Akad. Nauk SSSR **119** (1958), 1081–1084. (Russian)

4. M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. École Norm. Sup. (3) **71** (1954), 101–190.

5. H. Zassenhaus, *Ein Verfahren, jeder endlichen p-Gruppe eine Lie-Ring mit der Charakteristik p zuzuordnen*, Abh. Math. Sem. Hamburg Univ. **13** (1939), 200–207.

UNIVERSITY OF WISCONSIN

A GENERALIZATION OF THE CARTAN-BRAUER-HUA THEOREM

CHARLES J. STUTH

Let K be a division ring. Then K' will denote the multiplicative group of K. If S is a subset of K, then S^* will denote the division ring generated by S, and $C(S)$ will denote the centralizer of S in K. If G is a subgroup or a subdivision ring contained in K, then $Z(G)$ is the center of G. If x and y are elements of K', then $(x, y) = xyx^{-1}y^{-1}$ and $x^y = yxy^{-1}$. If b is an element of a group A, then $\mathrm{Cl}(A, b)$ will denote the group which is generated by the conjugate class of b in A.

A set is central in K if each of its elements is in $Z(K)$. A group G is n-subnormal in a group H if there are groups $G_1, \cdots, G_{n-1}$ such that $G = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = H$. A group G is subnormal in a group H if G is an n-subnormal subgroup of H for some n. A group G is invariant under a group H if $g^h \in G$ for all $g \in G$, all $h \in H$. A noncentral subgroup G is of type I if G is invariant under a noncentral subnormal subgroup of K'. K has the property P_n if for every subdivision ring H of K such that H' is invariant under a noncentral n-subnormal subgroup of K' it follows that H is central or $H = K$.

The Cartan-Brauer-Hua Theorem [2] states that a division ring has property P_0. Herstein and Scott [1] generalized this to P_1. Schenkman and Scott [5] extended the Cartan-Brauer-Hua Theorem by showing that a division ring has property P_n for all n if each of its subdivision rings which is invariant under a subnormal subgroup is normal in some subnormal subgroup of the division ring.

Theorem 1 of this paper shows that a division ring has property P_n for all n. Then results are developed from this concerning the subnormal subgroups of K' and more generally for the subgroups of type I in K'.

Received by the editors January 16, 1963.

LEMMA 0. *Let $x \in K$, $y \in K$, $(y, x) \neq 1$ commute with both x and y. Let $(y, 1+x) = y_1$, $y_i = (y, y_{i-1})$. If some $y_n = 1$, then x is algebraic over $Z(K)$.*

PROOF. This is Lemma 1 of [1] and follows immediately from Lemmas 1 and 2 of [4].

THEOREM 0. *Suppose every division ring has property P_{n-1}. Then if H is a subfield of K invariant under $G_n \lhd \cdots \lhd G_1 \lhd G_0 = K'$, where G_n is a noncentral subgroup of K', then $H \subset Z(K)$.*

PROOF. This is Theorem 2 of [1].

LEMMA 1. *Let $G_0, \cdots, G_m$ be groups such that $G_j \lhd G_{j-1}$ for $j = 1, \cdots, m$ with $h \in G_m$. Furthermore let $G(0, 0) = G_0$ and $G(i, 0) = \mathrm{Cl}(G(i-1, 0), h)$ for $i = 1, \cdots, m$. Then $G(m, 0)$ is m-subnormal in G_m.*

PROOF. For any group A with $x \in A$, $\mathrm{Cl}(A, x)$ is the smallest subgroup containing x which is normal in A. So if B is a normal subgroup of A with $x \in B$, then $\mathrm{Cl}(A, x) \lhd B$.

Let $G(i, i) = G_i$ for $i = 1, \cdots, m$. Assume inductively for all $i < k$ that $G(i-1, 0) \lhd \cdots \lhd G(i-1, i-1)$, where $G(i-1, j-1) = \mathrm{Cl}(G(i-2, j-1), h)$ for $j = 1, \cdots, i-1$. Let $G(k, j) = \mathrm{Cl}(G(k-1, j), h)$ for $j = 0, \cdots, k-1$. By the remark at the beginning of this proof $G(k, j) \lhd G(k-1, j-1)$ for $j = 1, \cdots, k-1$. By hypothesis $G(k, k) \lhd G(k-1, k-1)$. Then, again by the remark above, $G(k, j-1) \lhd G(k, j)$ for $j = 1, \cdots, k$.

LEMMA 2. *Let G be a group with $h \in G$. Let $M_0 = G$, $M_j = \mathrm{Cl}(M_{j-1}, h)$ for $j = 1, \cdots, m$. Then M_j is invariant under H for $j = 0, \cdots, m$ where H is the centralizer of h in G.*

PROOF. It is seen that $M_m \lhd \cdots \lhd M_0$. Assume inductively that M_{i-1} is invariant under H for some $i-1 < m$. Let $y \in H$, $g \in M_{i-1}$. Then $h^g \in M_i$ and $g^y \in M_{i-1}$ so that $(h^g)^y = g^y h (g^{-1})^y \in M_i$. Thus, $x^y \in M_i$ for all $x \in M_i$. Hence M_i is invariant under H.

Until Theorem 1 suppose that every division ring has property P_{n-1}, and that K is a division ring with a proper subdivision ring H which is not central in K and is invariant under $G_n \lhd \cdots \lhd G_1 \lhd G_0 = K'$, where G_n is a noncentral subgroup of K'.

LEMMA 3. *Let G be a noncentral n-subnormal subgroup of K'. For each noncentral $k \in K$ there exists $g \in G$ such that (g, k) is not central in K.*

PROOF. Suppose there is some noncentral $k \in K$ such that (g, k) is central for each $g \in G$. Then $k^g = (g, k)k$ so that $C(k^g) = C(k)$ for each

$g \in G$. Thus k and its G-conjugates commute, and they generate a noncentral subfield F which is invariant under G, contrary to Theorem 0.

LEMMA 4. *Let M be a noncentral subgroup of K' which is invariant under G_n. Then $M \cap G_n$ is not central in K.*

PROOF. Assume inductively that there exists noncentral $h \in M \cap G_{i-1}$ for some $i-1<n$. By Lemma 3 there exists $g \in G_n$ such that (h, g) is not central. Since $(h, g) \in M \cap G_i$, then $M \cap G_i$ is not central.

LEMMA 5. *Let h be a noncentral element in $H \cap G_n$. Then there exists $g \in G_n \backslash H$ such that (g, h) is not central.*

PROOF. If $G_n \subset H$, then by P_{n-1}, $K=(G_n)^* \subset H$, contrary to supposition on H. Therefore $G_n \not\subset H$, and there exists $x \in G_n \backslash H$.

Suppose (g, h) is central for all $g \in G_n \backslash H$. Let $w \in H \cap G_n$. Then $wx \in G_n \backslash H$. $(wx, h)=w(x, h)hw^{-1}h^{-1}=(x, h)(w, h)$. Since (wx, h) and (x, h) are central, then (w, h) is central. Therefore (g, h) is central for all $g \in G_n$, contrary to Lemma 3.

LEMMA 6.
(i) *$C(a) \subset H$ for all $a \in (H \cap G_n) \backslash Z(K)$;*
(ii) *$C(H)=Z(K)=Z(H)$.*

PROOF. Since H is invariant under G_n, then $Z(H)$ is a field invariant under G_n. By Theorem 0, $Z(H) \subset Z(K)$. If (i) is true, then $C(H) \subset H$, so that $Z(K) \subset C(H)=Z(H) \subset Z(K)$. Hence it suffices to prove (i).

By Lemma 4 there exists $h \in H \cap G_n$ which is not central. Suppose $C(h) \not\subset H$. Then there exists $y \in C(h) \backslash H$. Let $M_0 = K'$ and $M_i = \mathrm{Cl}(M_{i-1}, h)$ for $i=1, \cdots, n$. M_n is noncentral and n-subnormal in K'. By Lemma 1, $M_n \subset G_n$. By Lemma 2, M_n is invariant under $C(h)'$.

Let $g \in M_n$. Then $h^g \in H$ and $g^y \in M_n$. Hence $g^y h (g^{-1})^y = yghg^{-1}y^{-1} = c$ is in H. Let $z=1-y$. Since $z \in C(h) \backslash H$, then $zghg^{-1}z^{-1}=d$ is in H. Therefore $yh^g = cy$ and $zh^g = dz$. Adding, it follows that $h^g - d = (c-d)y$. If $c-d$ is not zero, then $y \in H$, a contradiction. Hence $d-c=0=h^g-d$, so that $h^g=c$. Therefore $y \in C(h^g)$. If w is an element of $C(h) \cap H$, then both y and $y-w$ are in $C(h) \backslash H \subset C(h^g)$, so that $w \in C(h^g)$. Hence $C(h) \subset C(h^g)$ for all $g \in M_n$.

If $b \in C(h)$ and $t=x^{-1} \in M_n$, then $b \in C(h^t)$, and thus $b^x \in C(h)$. Therefore $C(h)$ is invariant under M_n. Then $Z(C(h))$ is invariant under M_n and is a noncentral field in K because $h \in Z(C(h))$, contrary to Theorem 0. Hence $C(h) \subset H$.

REMARK. We now let $Z=Z(H)=Z(K)$ until Theorem 2.

LEMMA 7. *Let g be an element of the normalizer of H in K, $g \notin H$. Then $H \cap H^k = C(g) \cap H$, where $k = 1 + g$.*

PROOF. If $x \in C(g) \cap H$, then $x = x^k$, so $x \in H \cap H^k$. Hence $C(g) \cap H \subset H \cap H^k$.

Let $h \in H \cap H^k$. There exists $j \in H$ such that $h = j^k$. By hypothesis there exists $m \in H$ such that $h = m^g$. Then $hk = kj$ and $hg = gm$. Subtracting, it follows that $h - j = g(j - m)$. If $j - m$ is not zero, then $g \in H$, a contradiction. So $j - m = 0 = h - j$ and $h = m$. Then $h = h^g$ so that $h \in C(g)$. Therefore $H \cap H^k \subset C(g) \cap H$.

COROLLARY. *Under the hypothesis of Lemma 7, $G_n \cap H \cap H^k \subset Z$.*

PROOF. By Lemma 7, $G_n \cap H \cap H^k = G_n \cap H \cap C(g)$. If $h \in (G_n \cap H \cap C(g)) \backslash Z$, then $g \in C(h) \subset H$ by Lemma 6, a contradiction.

LEMMA 8. *There is no element in $(H \cap G_n) \backslash Z$ which is algebraic over Z.*

PROOF. Suppose $h \in (H \cap G_n) \backslash Z$ is algebraic over Z. By Lemma 5 there exists $g \in G_n \backslash H$. Let $Z(h)$ be the field generated by adjoining h to Z. Now h and h^g have the same minimal equation so there is an isomorphism between $Z(h)$ and $Z(h^g)$ (induced by $h \leftrightarrow h^g$) leaving Z elementwise invariant. By Corollary 2, p. 162 of [3], there exists $x \in H$ such that x induces the same inner automorphism as g does. Thus, $h^g = h^x$ from which $x^{-1}g \in C(h)$. By Lemma 6, $C(h) \subset H$ so that $x^{-1}g \in H$. Therefore, $g \in H$, a contradiction.

LEMMA 9. *There exists $g \in G_n \backslash H$ and $b \in (H \cap G_n) \backslash Z$ such that $b^{1+g} \in (H^{1+g} \cap G_n) \backslash Z$ and $b^x \in (H^x \cap G_n) \backslash Z$, where $x = (1+g)^{-1}$.*

PROOF. By Lemma 4 there exists $h_1 \in (H \cap G_n) \backslash Z$. By Lemma 5 there exists $g \in G_n \backslash H$ such that $(g, h_1) \in (H \cap G_n) \backslash Z$. Let $g_1 = (h_1)^{1+g}$, $g_i = (g, g_{i-1})$, $h_i = (g, h_{i-1})$ for $i = 2, \cdots, n$. By induction $h_i \in H \cap G_n$ for $i = 1, \cdots, n$.

Now $g_1 = (h_1)^{1+g} \in H^{1+g} \cap G_1$ because $h_1 \in G_n \subset G_1$. Suppose for some $j - 1 < n$ that $g_{j-1} = (h_{j-1})^{1+g} \in H^{1+g} \cap G_{j-1}$. Then $g_j = (g, g_{j-1}) = (g, (h_{j-1})^{1+g}) = (g, h_{j-1})^{1+g} = h_j^{1+g} \in H^{1+g}$. Now $g \in G_n \subset G_j$ and $(h_{j-1})^{1+g} \in G_{j-1}$, so that $(g, (h_{j-1}))^{1+g} \in G_j$. Therefore $g_j \in H^{1+g} \cap G_j$. By induction $g_i = h_i^{1+g} \in H^{1+g} \cap G_i$ for $i = 1, \cdots, n$.

Case I. h_n is not central. Then $g_n \in (H^{1+g} \cap G_n) \backslash Z$. The same argument above with $1 + g$ replaced by x will show that $(h_n)^x \in (H^{1+g} \cap G_n) \backslash Z$. Letting $b = h_n$, the lemma follows.

Case II. h_n is central. Then there exists an integer $k - 1 < n$ such that $h_{k-1} \in (H \cap G_n) \backslash Z$ and h_k is central. If $h_k = (g, h_{k-1}) = 1$, then

$g\in C(h_{k-1})\subset H$, contrary to Lemma 6. So $h_k\neq 1$ and commutes with h_{k-1} and g. Therefore $g_k\neq 1$ and g_k commutes with g_{k-1} and g.

Let $t_1=(g, g_{k-1}+1)$, $b_1=(g, h_{k-1}+1)$, $t_i=(g, t_{i-1})$, $b_i=(g, b_{i-1})$ for $i=2, \cdots, n+1$. By induction, $b_i\in H\cap G_i$ for $i=1, \cdots, n$. Then by induction, $t_i=(b_i)^{1+g}\in H^{1+g}\cap G_n$ for $i=1, \cdots, n$. If $t_m\in Z$ for some $m<n+1$, then $t_{m+1}=1$, and by Lemma 0, $g_{k-1}=(h_{k-1})^{1+g}$ is algebraic over Z; then h_{k-1} is algebraic over Z, contrary to Lemma 8. So t_i is not in Z for $i=1, \cdots, n$. Then $b_n^{1+g}=t_n\in(H^{1+g}\cap G_n)\backslash Z$. Since h_k commutes with h_{k-1} and g, then $(h_k)^x$ commutes with $(h_{k-1})^x$ and g. The argument of this paragraph with g_{k-1} replaced by $(h_{k-1})^x$ and $1+g$ replaced by x will prove that $(b_n)^x\in(H^x\cap G_n)\backslash Z$. Letting $b=b_n$, we are done.

THEOREM 1. *A division ring has property* P_k *for any non-negative integer* k.

PROOF. Assume inductively that every division ring has property P_{n-1}. Suppose K is a division ring without property P_n. Then there exist $H, G_1, G_2, \cdots, G_n$ such that H is a noncentral proper subdivision ring of K which is invariant under $G_n\triangleleft \cdots \triangleleft G_1\triangleleft G_0=K'$, where G_n is a noncentral subgroup of K'.

By Lemma 9 there exists $g\in G_n\backslash H$, $b\in H\cap G_n$ such that $b^{1+g}\in H^{1+g}\cap G_n$ and $b^x\in H^x\cap G_n$, where $x=(1+g)^{-1}$. Let $b^x=c$. Then $d=(b, b^{1+g})=(c, b)^{1+g}\in H^{1+g}$. Since $b^{1+g}\in G_n$, $b\in H\cap G_n$, then $d\in H\cap G_n$. Therefore $d\in Z$ by the corollary to Lemma 7.

If $d=1$, then $(c, b)=1$ and $c\in C(b)\subset H$ by Lemma 6. Hence $b=c^{1+g}\in H\cap G_n\cap H^{1+g}\subset Z$, contradiction. So $d\neq 1$. Let $a_1=(b, b^{1+g}+1)$, $d_1=(c, b+1)$, $a_i=(b, a_{i-1})$, $d_i=(c, d_{i-1})$ for $i=2, \cdots, n+1$. Also let $G_n=G_{n+1}$. By induction $a_i=(d_i)^{1+g}\in H^{1+g}\cap G_i$ for $i=1, \cdots, n+1$. Now $a_{n+1}=(b, a_n)\in H$, so that $a_{n+1}\in H^{1+g}\cap G_n\cap H\subset Z$. Therefore $(b, a_{n+1})=1$. Since $d\in Z$, then d commutes with b and b^{1+g}. By Lemma 0, b^{1+g} is algebraic over Z, and thus b is algebraic over Z, contrary to Lemma 8.

THEOREM 2. *Let* G *be a noncentral subnormal subgroup of* K'.

(i) *For each noncentral* $k\in K$ *there exists* $g\in G$ *such that* (g, k) *is not central*;

(ii) $C(G)=Z(K)$.

PROOF. Lemma 3 and Theorem 1 give (i). Then (ii) follows from (i).

THEOREM 3. *Let* $x\in K\backslash Z(K)$, G *a noncentral subnormal subgroup of* K', *and* M *the conjugate class of* x *in* G. *Then* $M^*=K$.

PROOF. By Theorem 1.

Theorem 4. *K' has no proper noncentral subnormal solvable subgroups.*

Proof. Suppose G is a proper noncentral subnormal solvable subgroup of K'. There exist groups $G_1, \cdots, G_n$ such that $\{1\} = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$, where G_i is the commutator subgroup of G_{i-1} for $i = 1, \cdots, n$. Now G_i is a subnormal subgroup of K' for $i=0, \cdots, n$. By induction using Theorem 2, G_i is not central for $i=1, \cdots, n$. But $G_n = \{1\}$ is central, a contradiction.

Theorem 5. *Let A and B be noncentral subnormal subgroups of K'. Then so is $A \cap B$.*

Proof. Without loss of generality assume A and B are both n-subnormal in K'. Let $A = A_{2n} = \cdots = A_n \triangleleft \cdots \triangleleft A_1 \triangleleft A_0 = K'$ and $B = B_{2n} = \cdots = B_n \triangleleft \cdots \triangleleft B_1 \triangleleft B_0 = K'$. Assume inductively that for some $i < 2n$ that $A_j \cap B_k$ is noncentral i-subnormal subgroup of K', for all non-negative integers j and k such that $j+k=i$. Consider $A_g \cap B_h$ for positive integers g and h with $g+h=i+1$. Since $g+(h-1) = (g-1)+h=i$, then there exists $x \in A_g \cap B_{h-1}$ which is not central. By Theorem 2 there exists an element $y \in A_{g-1} \cap B_h$ such that (x, y) is not central. Since $A_g \cap B_h$ is normal in both $A_g \cap B_{h-1}$ and $A_{g-1} \cap B_h$, then $(x, y) \in A_g \cap B_h$. So $A_g \cap B_h$ is a noncentral $(i+1)$-subnormal subgroup of K'. By assumption $A_{i+1} \cap B_0 = A_{i+1}$ and $A_0 \cap B_{i+1} = B_{i+1}$ are noncentral $(i+1)$-subnormal subgroups of K'. By induction $A \cap B = A_n \cap B_n$ in a noncentral $2n$-subnormal subgroup of K'.

Theorem 2 (ii) generalizes Huzurbazar [**4**], and Theorem 3 generalizes Hua [**2**]. Theorems 4 and 5 extend Scott's results in [**6**]. The following theorem extends the results about noncentral subnormal subgroups of K' to subgroups of type I.

Theorem 6. *Let H and L be subgroups of K', both of type* I. *Then*

(i) $C(H) = Z(K)$;

(ii) *for each noncentral $k \in K$ there is an $h \in H$ such that (k, h) is not central*;

(iii) *H is not solvable*;

(iv) $H^* = K$;

(v) *$H \cap L$ is of type* I.

Proof. There exists noncentral groups $G_1, \cdots, G_n$ such that H is invariant under $G = G_n \triangleleft \cdots \triangleleft G_1 \triangleleft K'$. By Lemma 4, $H \cap G$ is noncentral and also $H \cap G \triangleleft G$. Then (i)–(iii) follow from Theorems 2 and 4 via $H \cap G$. H^* is invariant under G, hence $H^* = K$ by Theorem 1.

Let M be a noncentral subnormal subgroup of K' such that L is invariant under M. By Theorem 5, $G\cap M$ is a noncentral subnormal subgroup of K'. $H\cap L$ is invariant under $G\cap M$. By Lemma 4, $L\cap M$ is noncentral so that by Theorem 5, $(H\cap G)\cap(L\cap M)$ is not central, hence $H\cap L$ is not central.

THEOREM 7. *Let H be a subgroup of type* I *in K'. Then H is not finite.*

PROOF. Suppose H is finite of minimum order. There exists a noncentral group G such that H is invariant under G. By Lemma 4, $H\cap G$ is noncentral subnormal in K'. $H\subset G$ since H is minimal. Hence H is a noncentral n-subnormal subgroup of K', where n is minimal, say $H=H_n\triangleleft H_{n-1}\triangleleft \cdots \triangleleft H_1\triangleleft H_0=K'$.

Case I. $n=1$. Now there exists $x\in H\backslash Z(K)$. Since $H\triangleleft K'$, then x has a finite number of conjugates in K', contrary to Theorem 4 of [6].

Case II. $n>1$. Then by the minimality of n, there is $y\in H_{n-2}$ such that $H^y\neq H$. $H^y\triangleleft H_{n-1}$ and H^y is not central, so by Theorem 5 $H\cap H^y$ is noncentral. Also $H\cap H^y\triangleleft H_{n-1}$. This contradicts the minimality of H.

The author wishes to express his appreciation to the National Science Foundation for its support and to Professor W. R. Scott for his suggestions, especially on Theorem 7.

BIBLIOGRAPHY

1. I. N. Herstein and W. R. Scott, *Subnormal subgroups of division rings*, Canad. J. Math. **15** (1963), 80–83.

2. L. K. Hua, *Some properties of a sfield*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 533–537.

3. Nathan Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ. Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.

4. M. S. Huzurbazar, *The multiplicative group of a division ring*, Dokl. Akad. Nauk SSSR **131** (1960), 1268–1271.

5. Eugene Schenkman and W. R. Scott, *A generalization of the Cartan-Brauer-Hua theorem*, Proc. Amer. Math. Soc. **11** (1960), 396–398.

6. W. R. Scott, *On the multiplicative group of a division ring*, Proc. Amer. Math. Soc. **8** (1957), 303–305.

UNIVERSITY OF KANSAS AND
EAST TEXAS STATE COLLEGE