A NOTE ON GROUPS OF PRIME POWER EXPONENT SATISFYING AN ENGEL CONGRUENCE

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1. In [3] Kostrikin has proved that a Lie algebra of characteristic p , p prime, satisfying the n th Engel condition, $n \leq p$, is locally nilpotent. Using this result he has shown that in a group of exponent p generated by r elements, r finite, one has $G_m = G_{m+1}$ (where G_i denotes the i th term of the lower central series of G) and m is dependent only on p and r . In this note we shall apply Kostrikin's result to groups of exponent p^α .

2. Let us denote by (x, y) the group commutator $x^{-1}y^{-1}xy$. We define the symbol $(x, y; n)$ recursively by

$$(x, y; 1) = (x, y), \quad (x, y; n+1) = ((x, y; n), y),$$

that is, $(x, y; n)$ is a left normed commutator of weight $n+1$. A group G is said to satisfy the n th Engel congruence if $(x, y; n) \equiv 1 \pmod{G_{n+2}}$ for all $x, y \in G$. In a group of exponent p the $(p-1)$ st Engel congruence holds (P. Hall, [1]).

3. If $G = H_1 \supseteq H_2 \supseteq H_3 \supseteq \dots$ is a descending central series for G having the property that $(H_i, H_j) \subseteq H_{i+j}$ for all i and j then one can construct, in the canonical fashion, the Lie ring

$$L(H_i) = \sum (H_i/H_{i+1}), \quad \text{direct sum}$$

(cf. Lazard, [4], and Higman, [2]). In particular, if $\{G_i\}$ is the lower central series of G we obtain the Lie ring $L(G_i)$ used by Higman, [2], and Kostrikin, [3]. It has been shown by Higman, [2], that if G satisfies the n th Engel congruence then the relation $n!xy^n = 0$ is satisfied for all $x, y \in L(G_i)$.

4. Since $G_i \subseteq H_i$ for all i , we have $G_i H_{i+1}/H_{i+1}$ a subgroup of H_i/H_{i+1} . Hence

$$L^*(H_i) = \sum (G_i H_{i+1}/H_{i+1}), \quad \text{direct sum,}$$

is a subring of $L(H_i)$. For $x \in G_i$ and $y \in G_j$ we have $(x, y) \in G_{i+j} \subseteq G_{i+j} H_{i+j+1}$ and so it follows that $L^*(H_i)$ is a Lie subring of $L(H_i)$. Moreover, the mappings $\phi_i: G_i/G_{i+1} \rightarrow G_i H_{i+1}/H_{i+1}$ given by $xG_{i+1} \rightarrow xH_{i+1}$ are homomorphisms. The mapping $\phi = \sum \phi_i$ is then easily

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seen to be a homogeneous Lie ring homomorphism of $L(G_i)$ onto $L^*(H_i)$. Thus $L^*(H_i)$ is a subring of $L(H_i)$ and a homomorphic image of $L(G_i)$.

5. We can now state and prove our result.

THEOREM. *Let G be a finitely generated group of exponent p^α , p prime, such that*

- (i) *G satisfies the $(p-1)$ st Engel congruence,*
- (ii) *for some integer $m > 0$, G_m has exponent p .*

Then there exists an integer n such that $G_n = G_{n+1}$.

PROOF. In [5] Zassenhaus shows that the sequence of subgroups $H_k = \prod G_i^{p^j}$, the product taken over all i and j such that $ip^j \geq k$, form a descending central series for G satisfying the conditions $(H_r, H_s) \subseteq H_{r+s}$ and $H_r^p \subseteq H_{rp} \subseteq H_{p+1}$. Hence $L(H_i)$ is a Lie algebra of characteristic p and, hence, so is $L^*(H_i)$. Since the $(p-1)$ st Engel congruence holds in G , the relation $(p-1)!xy^{p-1} = 0$ holds in $L(G_i)$ and hence also in the homomorphic image $L^*(H_i)$. Since $L^*(H_i)$ has characteristic p , this implies that the $(p-1)$ st Engel condition, $xy^{p-1} = 0$, holds in $L^*(H_i)$. Since G is finitely generated, $L^*(H_i)$ is also finitely generated and so, by Kostrikin's result, $L^*(H_i)$ is nilpotent. Thus for some integer N we have $G_n \subseteq H_{n+1}$ or

$$(1) \quad G_n \subseteq G_{n+1} G_n^p G_{n-1}^p \cdots = \prod G_i^{p^j}$$

for all $n \geq N$. By taking n large enough we can ensure that in (1) each term G_k with $k < m$ occurs with exponent $\geq p^\alpha$. Then (1) will reduce to $G_n \subseteq G_{n+1}$, that is, $G_n = G_{n+1}$.

6. This result is not as strong as one might hope for. We would like to have the index n depend only on p^α , m and the number of generators of G . However it is not possible to assert the existence of a universal group G generated by r elements and satisfying the conditions of the theorem and having any other such group as a homomorphic image. Since the sequence $\{H_i\}$ of Zassenhaus is the most rapidly descending central series such that H_i/H_{i+1} has exponent p it seems unlikely that a stronger result can be obtained using the method of this note unless one can strengthen Kostrikin's theorem.

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A GENERALIZATION OF THE CARTAN-BRAUER-HUA THEOREM

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Let K be a division ring. Then K' will denote the multiplicative group of K . If S is a subset of K , then S^* will denote the division ring generated by S , and $C(S)$ will denote the centralizer of S in K . If G is a subgroup or a subdivision ring contained in K , then $Z(G)$ is the center of G . If x and y are elements of K' , then $(x, y) = xyx^{-1}y^{-1}$ and $x^y = yxy^{-1}$. If b is an element of a group A , then $\text{Cl}(A, b)$ will denote the group which is generated by the conjugate class of b in A .

A set is central in K if each of its elements is in $Z(K)$. A group G is n -subnormal in a group H if there are groups $G_1, \dots, G_{n-1}$ such that $G = G_n \triangleleft G_{n-1} \triangleleft \dots \triangleleft G_1 \triangleleft G_0 = H$. A group G is subnormal in a group H if G is an n -subnormal subgroup of H for some n . A group G is invariant under a group H if $g^h \in G$ for all $g \in G$, all $h \in H$. A non-central subgroup G is of type I if G is invariant under a noncentral subnormal subgroup of K' . K has the property P_n if for every subdivision ring H of K such that H' is invariant under a noncentral n -subnormal subgroup of K' it follows that H is central or $H = K$.

The Cartan-Brauer-Hua Theorem [2] states that a division ring has property P_0 . Herstein and Scott [1] generalized this to P_1 . Schenkman and Scott [5] extended the Cartan-Brauer-Hua Theorem by showing that a division ring has property P_n for all n if each of its subdivision rings which is invariant under a subnormal subgroup is normal in some subnormal subgroup of the division ring.

Theorem 1 of this paper shows that a division ring has property P_n for all n . Then results are developed from this concerning the subnormal subgroups of K' and more generally for the subgroups of type I in K' .

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