$2(2d+1)$ is not a kth power residue modulo p. Since n was arbitrary then $\Lambda(k, 4)=\infty$. This proves the theorem.

References

1. E. Kummer, Abh. K. Akad. Wiss. Berlin (1859).
2. D. H. and E. Lehmer, *On runs of residues*, Proc. Amer. Math. Soc. **13** (1962), 102–106.
3. W. H. Mills, *Characters with preassigned values*, Canad. J. Math. **15** (1963), 169–171.

Bell Telephone Laboratories

ON DECOMPOSITIONS OF PARTIALLY ORDERED SETS

E. S. WOLK

1. Introduction. Let P be a set which is partially ordered by a relation $\leqq$. A *decomposition* $\mathfrak{D}$ of P is a family of mutually disjoint nonempty chains in P such that $P=\bigcup\{C: C\in\mathfrak{D}\}$. Two elements x, y of P are *incomparable* if and only if $x\not\leqq y$ and $y\not\leqq x$. A *totally unordered* set in P is a subset in which every two different elements are incomparable. We denote the cardinal number of a set S by $|S|$.

Dilworth [1] has proved the following well-known decomposition theorem.

Theorem 1 (Dilworth). *Let P be a partially ordered set, and suppose that n is a positive integer such that*

$$n = \max\{|A| : A \text{ is a totally unordered subset of } P\}.$$

Then there is a decomposition $\mathfrak{D}$ of P with $|\mathfrak{D}|=n$.

It is natural to ask whether, in this theorem, the positive integer n may be replaced by an infinite cardinal number. However, the theorem is no longer valid in this case, as is shown by an example in [3] which is due in essence of Sierpinski [2]. In this example P is a set of pairs which represents a 1-1 mapping from ω_1, the first uncountable ordinal, into the real numbers. $(x_1, y_1)\leqq(x_2, y_2)$ is defined by: $x_1\leqq x_2$ (as ordinals) and $y_1\leqq y_2$ (as real numbers). The purpose of this note is to show that a similar idea leads, given any infinite cardinal k, to

Received by the editors December 28, 1962.

a construction of a partially ordered set in which all totally unordered subsets are finite but every decomposition is of power k. We also give an application of this result to the theory of graphs.

In the following we identify cardinals with initial ordinals. If C is any chain and $B\subseteq C$, we shall say that B is *cofinal* in C if and only if for each $x\in C$ there exists $y\in B$ with $x\leqq y$.

2. Main result.

THEOREM 2. *Let k be any infinite cardinal. Let $Q(k)=k\times k$, and let a partial ordering on $Q(k)$ be defined by: $(x_1, y_1)\leqq(x_2, y_2)$ if and only if $x_1\leqq x_2$ and $y_1\leqq y_2$. Then*

(i) *every totally unordered subset of $Q(k)$ is finite, and*

(ii) *every decomposition of $Q(k)$ is of power k.*

PROOF. (i) If $x_1=x_2$, then (x_1, y_1) and (x_2, y_2) are not incomparable. Hence in a totally unordered subset of $Q(k)$ first coordinates of different members are different; they are also well ordered by $\leqq$. Therefore, if there is an infinite totally unordered subset it would include a sequence $(x_1, y_1), \cdots, (x_n, y_n), \cdots$ in which $x_1<x_2<\cdots<x_n<x_{n+1}<\cdots$. If $y_n\leqq y_{n+1}$ we would have $(x_n, y_n)\leqq(x_{n+1}, y_{n+1})$, and hence $y_1>y_2>\cdots>y_n>y_{n+1}>\cdots$; but this is impossible as the y_n's are well ordered by $\leqq$.

(ii) Since $|Q(k)|=k$ every decomposition of $Q(k)$ is of power $\leqq k$. Hence it suffices to show that every decomposition is of power $\geqq k$. First assume that k is regular. For $v<k$, let us define $L_v=k\times\{v\}=\{(a, v): a<k\}$. If C is a chain in $Q(k)$ such that C is cofinal in L_v and $v<v'$, then $C\cap L_{v'}=\emptyset$. For if $(a, v')\in C$, then there is an a' such that $a'>a$ and $(a', v)\in C$, but (a, v') and (a', v) are incomparable. In particular, no chain is cofinal in both L_v and $L_{v'}$ if $v\neq v'$.

Let $\mathfrak{D}$ be any decomposition of $Q(k)$. If for every $v<k$ there is a C in $\mathfrak{D}$ such that C is cofinal in L_v, then it follows by the observation just made that $|\mathfrak{D}|\geqq k$. If on the other hand there is a v such that no C in $\mathfrak{D}$ is cofinal in L_v, it follows from the regularity of k and from the fact that $\bigcup\{C: C\in\mathfrak{D}\}\supseteq L_v$, that $|\mathfrak{D}|\geqq k$.

Now if k is any infinite cardinal and $\mathfrak{D}$ is a decomposition of $Q(k)$, then $\{C\cap Q(h): C\in\mathfrak{D}\}$ is a decomposition of $Q(h)$ for every cardinal $h\leqq k$. Hence, for every regular cardinal h which is $\leqq k$, we have $|\mathfrak{D}|\geqq h$, and therefore $|\mathfrak{D}|\geqq k$. This completes the proof.

3. An application to graph theory.

Let G be a set, and let G^2 denote the set of all two-element subsets of G. By a *graph* we mean a pair (G, R), where G is a set and $R\subseteq G^2$. If the unordered pair $\{x, y\}$ is an element of R, we write xRy: if this is not the case, we write $x\tilde{R}y$.

A subset H of G is *complete* if and only if xRy for all $x\in H$, $y\in H$. A subset H of G is *independent* if and only if $x\tilde{R}y$ for all $x\in H$, $y\in H$. A *decomposition* of a graph (G, R) is a family of mutually disjoint nonempty independent subsets of G whose union is G. For any graph (G, R), we define

$$d(G) = \text{l.u.b.}\{\,|H| : H \text{ is a complete subset of } G\},$$

$$c(G) = \min\{\,|\mathfrak{D}| : \mathfrak{D} \text{ is a decomposition of } (G, R)\}.$$

It is clear that $d(G)\leqq c(G)$ for all graphs (G, R).

Zykov [**4**, Theorem 8] has shown that, given any positive integers d_0 and c_0 with $d_0\leqq c_0$, there is a graph (G, R) such that $d(G)=d_0$ and $c(G)=c_0$. Using Theorem 2, we now show that this result may be extended to infinite cardinals. We shall prove

THEOREM 3. *Given any infinite cardinal numbers k and m with $k\leqq m$, there exists a graph (G, R) such that $d(G)=k$ and $c(G)=m$.*

PROOF. Let P be any partially ordered set. If x, $y\in P$, define xRy if and only if x and y are incomparable with respect to the partial order in P. We call the graph (P, R) the *incomparability graph* of the partially ordered set P.

Now, given the cardinal numbers k and m, assume first that $k=\aleph_0$. Then the incomparability graph of the partially ordered set $Q(m)$ satisfies the required conditions. If $k>\aleph_0$, we adjoin to $Q(m)$ a set A of mutually incomparable elements with $|A|=k$. We define $r\leqq s$ for all $r\in A$ and $s\in Q(m)$, and we retain the previously defined partial order within $Q(m)$. The reader may now verify that the incomparability graph of the partially ordered set $A\cup Q(m)$ satisfies the requirements of the theorem.

REFERENCES

1. R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. (2) **51** (1950), 161–166.

2. W. Sierpinski, *Sur un problème de la théorie des relations*, Ann. Scuola Norm. Sup. Pisa **2** (1933), 285–287.

3. E. S. Wolk, *The comparability graph of a tree*, Proc. Amer. Math. Soc. **13** (1962), 789–795.

4. A. A. Zykov, *On some properties of linear complexes*, Amer. Math. Soc. Transl. no. 79, 1952.

UNIVERSITY OF CONNECTICUT