THE COEFFICIENTS OF THE RECIPROCAL OF A BESSEL FUNCTION[1]

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Put

$$\left\{\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!n!}\right\}^{-1}=\sum_{n=0}^{\infty}\frac{\omega_n x^n}{n!n!}.$$

This is equivalent to

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}^2\omega_r=\begin{cases}1 & (n=0)\\ 0 & (n>0).\end{cases}\tag{1}$$

In a letter to the author, J. Riordan has raised the question whether the ω_n can satisfy a recurrence of order independent of n. We shall show that the ω_n cannot satisfy a recurrence order k, where k is independent of n, with polynomial coefficients. More precisely we show that the assumption

$$\sum_{j=0}^{k}A_j(n)\omega_{n+j}=0\qquad(n>N),\tag{2}$$

where the $A_j(n)$ are polynomials in n with complex coefficients and k, N are fixed, leads to a contradiction.

Since it is no more difficult, we consider the following more general problem. Put

$$\left\{\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!\Gamma(\nu+n+1)}\right\}^{-1}=\sum_{n=0}^{\infty}\frac{\omega_n(\nu)x^n}{n!\Gamma(\nu+n+1)}.\tag{3}$$

This is equivalent to

$$\sum_{r=0}^{n}(-1)^r\binom{\nu+n}{r}\binom{\nu+n}{n-r}\omega_r(\nu)=\begin{cases}1 & (n=0)\\ 0 & (n>0).\end{cases}\tag{4}$$

We assume that ν is not a negative integer; then it is clear that the $\omega_n(\nu)$ are uniquely determined by (3) or (4).

Now assume that the $\omega_n(\nu)$ satisfy the recurrence

$$\sum_{j=0}^{k}A_j(n,\nu)\omega_{n+j}(\nu)=0\tag{5}$$

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for all $n > N$, where the $A_j(n, \nu)$ are polynomials in n with complex coefficients and k, N are fixed. Put

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!\Gamma(\nu + n + 1)},$$

$$g(x) = \frac{1}{f(x)} = \sum_{n=0}^{\infty} \frac{\omega_n(\nu) x^n}{n!\Gamma(\nu + n + 1)}.$$

Now if $P(x)$ is an arbitrary polynomial with constant coefficients, it is evident that

$$P(xD)g(x) = \sum_{n=0}^{\infty} P(n) \frac{\omega_n(\nu) x^n}{n!\Gamma(\nu + n + 1)},$$

where $D = d/dx$; moreover since

$$D^j g(x) = \sum_{n=0}^{\infty} \frac{\omega_{n+j}(\nu) x^n}{n!\Gamma(\nu + n + j + 1)},$$

it follows that

$$P(xD) \cdot D^j g(x) = \sum_{n=0}^{\infty} \frac{P(n)}{(\nu + n + 1)_j} \frac{\omega_{n+j}(\nu) x^n}{n!\Gamma(\nu + n + 1)}. \tag{6}$$

If we multiply both sides of (5) by

$$\frac{x^n}{n!\Gamma(\nu + n + 1)}$$

and sum over all $n > N$ we get

$$\sum_{j=0}^{k} \sum_{n=0}^{\infty} A_j(n, \nu) \frac{\omega_{n+j}(\nu) x^n}{n!\Gamma(\nu + n + 1)} = C(x), \tag{7}$$

where $C(x)$ is a polynomial in x of degree $\leq N$. Repeated differentiation of (7) leads to an equation of the same kind in which the right member vanishes.

Comparison of (7) with (6) shows that $g(x)$ satisfies a differential equation of the form

$$\sum_{j=0}^{m} B_j(x, \nu) D^{m-j} g(x) = 0, \tag{8}$$

where the $B_j(x, \nu)$ are polynomials in x. The order m depends upon the degree of the $A_j(n, \nu)$. We may assume that

$$B_0(x, \nu) \neq 0. \tag{9}$$

In the next place since $g(x)=1/f(x)$, we have

$$(10)\qquad \begin{aligned} g'(x) &= -\frac{f'(x)}{f^2(x)}, \qquad g''(x) = -\frac{f''(x)}{f^2(x)} + \frac{2(f'(x))^2}{f^3(x)}, \\ g'''(x) &= -\frac{f'''(x)}{f^2(x)} + 6\frac{f'(x)f''(x)}{f^3(x)} - 6\frac{(f'(x))^3}{f^4(x)}, \end{aligned}$$

and so on. Making use of (10) we may replace (8) by a differential equation in $f(x)$.

For simplicity we shall assume $m=3$; the method is however quite general. We find that

$$(11)\qquad \begin{aligned} &B_0\{-f^2(x)f'''(x) + 6f(x)f'(x)f''(x) - 6(f'(x))^3\} \\ &\quad + B_1\{-f^2(x)f''(x) + 2f(x)(f'(x))^2\} - B_2f^2(x)f'(x) + B_3f^3(x) = 0, \end{aligned}$$

where $B_j=B_j(x, \nu)$. Now, on the other hand, we have

$$xf''(x) + (\nu+1)f'(x) + f(x) = 0,$$

so that

$$xf'''(x) + (\nu+2)f''(x) + f'(x) = 0.$$

We may eliminate $f''(x)$ and $f'''(x)$ in (11); there results an equation of the form

$$(12)\qquad \begin{aligned} &C_0(x, \nu)(f'(x))^3 + C_1(x, \nu)(f'(x))^2f(x) \\ &\qquad\qquad + C_2(x, \nu)f'(x)f^2(x) + C_3(x, \nu)f^3(x) = 0, \end{aligned}$$

where $C_j(x, \nu)$ are polynomials in x. Moreover, by (9) $C_0(x, \nu) = -6B_0(x, \nu)\neq 0$.

It therefore follows from (12) that $f'(x)/f(x)$ is an algebraic function of x. However, since $f(x)$ has infinitely many zeros, it follows that the logarithmic derivative $f'(x)/f(x)$ has infinitely many poles and therefore cannot be an algebraic function.

We have proved the following

THEOREM. *Let ν be an arbitrary complex number not equal to a negative integer and define $\omega_n(\nu)$ by means of (3). Then $\omega_n(\nu)$ cannot satisfy a recurrence*

$$\sum_{j=0}^{k} A_j(n, \nu)\omega_{n+j}(x) = 0 \qquad (n > N),$$

where the $A_j(n, \nu)$ are polynomials in n with complex coefficients and k, N are fixed.

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