NOTE ON THE OCCURRENCE OF DIRECT FACTORS IN GROUPS

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J. Wiegold demonstrated [1] that the groups for which each normal subgroup is a direct factor are precisely the direct products (in the restricted sense) of simple groups. In this note we study the groups for which each proper normal subgroup is contained in a proper direct factor and also the groups for which each nonunit normal subgroup contains a nonunit direct factor. The result of Wiegold stated above may be regarded as a corollary to our Theorem 1. Our terms and notation are from [2].

THEOREM 1. *Each proper normal subgroup of group G is contained in a proper direct factor of G if and only if G is a direct product (restricted) of simple groups.*

PROOF. Suppose that each proper normal subgroup of G is contained in a proper direct factor of G . Let $[S_\alpha]_{\alpha \in A}$ be the set of all simple normal subgroups of G and let K be the subgroup generated by this set. We will show that $G=K$. Suppose $h \in G \setminus K$. Let M be a normal subgroup of G maximal subject to: $h \notin M \supset K$. Let M' be a proper direct factor of G containing M . Let $G=M' \times S$. If h were in M' we would have the contradiction of the maximality of M : $h \notin M \times S$. Thus $h \notin M'$ and by the maximality of M , $M'=M$. Let N be a non-unit normal subgroup of S . Since the only direct factors containing M are M and G we have $M \times N=G$ and consequently $N=S$. We have the contradiction: S is a simple normal subgroup of G which is not contained in K . We conclude that $G=K$. By Zorn's lemma there is a subset B of A for which $G=\prod_{\alpha \in B} S_\alpha$.

Conversely, if $G=\prod_{\alpha \in B} S_\alpha$ where each S_α is simple and if N is a normal subgroup of G then the family of those subsets $C \subset B$ for which the product $N \times \prod_{\alpha \in C} S_\alpha$ is direct is closed under the operation of forming unions of inclusion chains. By Zorn's lemma there is a subset $D \subset B$ which is maximal subject to the requirement that the product $N \times \prod_{\alpha \in D} S_\alpha$ be direct. By the maximality of D and the simplicity of each S_α we have $S_\alpha \cap N \times \prod_{\alpha \in D} S_\alpha = S_\alpha$ for every $\alpha \in B$. Consequently $G=N \times \prod_{\alpha \in D} S_\alpha$ which shows that N is a direct factor of G .

THEOREM 2. *A group G is isomorphic to a subdirect product of a*

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family of simple groups if and only if for each nonunit normal subgroup N of G there is a proper normal subgroup N' of G for which $NN' = G$.

PROOF. Suppose G is a subdirect product of simple groups. Suppose N is a nonunit normal subgroup of G . Let $1 \neq x \in N$ and let N' be a maximal normal subgroup of G not containing x . The existence of such a maximal normal subgroup follows from the fact that each of the simple groups under discussion is a homomorphic image of G . We have $NN' = G$.

Conversely, let G be a group for which for each nonunit normal subgroup N of G there is a proper normal subgroup N' of G for which $NN' = G$. Suppose $1 \neq x \in G$. Let N be the normal subgroup of G generated by x and let N' be a proper normal subgroup of G for which $NN' = G$. Let M be a normal subgroup of G maximal subject to: $x \notin M \supset N'$. Such an M must be a maximal normal subgroup of G . Thus each nonunit element of G lies outside a maximal normal subgroup of G , and G is therefore isomorphic to a subdirect product of a family of simple groups.

COROLLARY. *If each nonunit normal subgroup of a group G contains a nonunit direct factor of G then G is isomorphic to a subdirect product of a family of simple groups.*

PROOF. For each nonunit normal subgroup N of G we have a direct decomposition $G = H \times N'$ where $H \subset N$ and $N' \neq G$. Then $NN' = G$. By Theorem 2, G is isomorphic to a subdirect product of simple groups.

Our final theorem has discouraged us from trying to determine the structure of those groups G having the property that each nonunit normal subgroup contains a nonunit direct factor. The proof is based on the following theorem which appears, for instance, as Theorem 16 in [3]: *Any group can be embedded in a noncyclic simple group.*

THEOREM 3. *Any group A can be embedded as a semidirect factor in a group G having the property that each nonunit normal subgroup of G contains a nonunit direct factor of G .*

PROOF. Let A be a group. Embed A in a noncyclic simple group S . Let $P = \prod^* S_i$ (unrestricted direct product, i a natural number) where $S_i = S$ for each i . Let $W = \prod S_i$ (restricted direct product) and regard W as a subset of P in the natural way. For each $b \in A$ let $\bar{b}$ be that element of P which has b as its i th component for each i . The elements $\{\bar{b} | b \in A\}$ form a subgroup $\bar{A}$ of P which is isomorphic to A .

Identify $\bar{A}$ with A . Let G be the subgroup of P generated by W and A . W is normal in G and $W \cap A = 1$. Thus $G = WA$ is a semidirect product of W and A . Let N be any nonunit normal subgroup of G . Let $1 \neq x \in N$. Let i be such that the i th component, x_i , of x is not 1. Since S_i is a centerless group there is a $y_i \in S_i$ for which $x_i y_i \neq y_i x_i$. Let y be the element of P which has i th component y_i and j th component 1 for $j \neq i$. $y \in W \subset G$. Then $1 \neq x^{-1} y^{-1} x y \in N \cap S_i$. Since S_i is simple $N \supset S_i$ which is a direct factor of G .

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