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ISOMETRIC ISOMORPHISMS OF MEASURE ALGEBRAS

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1. Introduction. We shall prove that two locally compact topological groups with isometrically isomorphic measure algebras are themselves isomorphic. This is similar to a result of Wendel (see [2]) proved for group algebras. We first characterize the absolutely continuous measures on a group in terms of the metric and algebra properties of the measure algebra. It then follows that if two locally compact groups have isometrically isomorphic measure algebras then they have isometrically isomorphic group algebras, and so our result follows from Wendel's theorem.

2. Singularity in terms of norm. Throughout this section X will denote a locally compact Hausdorff space. The set $M(X)$ of bounded complex Radon measures on X forms a Banach space (see [1, p. 57]).

LEMMA 1. *If α, β are two complex numbers then*

$$|\alpha + \beta| + |\alpha - \beta| \leq 2|\alpha| + 2|\beta|,$$

equality occurring if and only if $\alpha = 0$ or $\beta = 0$.

PROOF. This result follows from the elementary inequalities $|\alpha \pm \beta| \leq |\alpha| + |\beta|$ by considering the occasions in which these are both equalities.

THEOREM 1. *Let $\mu_1, \mu_2 \in M(X)$. Then μ_1 and μ_2 are mutually singular if and only if*

$$(1) \quad \|\mu_1 + \mu_2\| + \|\mu_1 - \mu_2\| = 2\|\mu_1\| + 2\|\mu_2\|.$$

PROOF. Put $\nu = |\mu_1| + |\mu_2|$. Both μ_1 and μ_2 are absolutely continuous with respect to ν , and so there are ν -integrable functions f_1, f_2 on X , such that $\mu_i(E) = (f_i \cdot \nu)(E) = \int_E f_i(t) d\nu(t)$, $i = 1, 2$, for each Borel set E in X . We have

$$\begin{aligned}
& 2\|\mu_1\| + 2\|\mu_2\| - \|\mu_1 + \mu_2\| - \|\mu_1 - \mu_2\| \\
&= \int_x 2|f_1(t)| + 2|f_2(t)| - |f_1(t) + f_2(t)| - |f_1(t) - f_2(t)| d\nu(t).
\end{aligned}$$

By Lemma 1, the integrand is non-negative so that (1) holds if and only if $\{x; x \in X, f_1(x) \neq 0, f_2(x) \neq 0\}$ is of ν measure zero. This implies the result.

3. Absolute continuity in terms of norm. Throughout this section G will denote a locally compact topological group. In this case $M(G)$ becomes a Banach algebra if we take convolution as multiplication (see [3]). The set of measures in $M(G)$ which are of unit norm and have one point support will be denoted by H .

DEFINITION 1. Let $\mu, \nu \in M(G)$. We say that μ and ν are of *different type* if μ and $\gamma * \nu$ are mutually singular for all $\gamma \in H$.

We shall say that a measure μ is singular if μ is singular with respect to Haar measure on G . This definition is not ambiguous since both left and right Haar measure give rise to the same class of singular measures. We will denote left Haar measure by λ .

LEMMA 2. Let μ_1, μ_2 be nonsingular elements of $M(G)$. Then μ_1 and μ_2 are not of different type.

PROOF. We can write $\mu_i = \nu_i + f_i \cdot \lambda$, where ν_i is singular and $f_i \in L_1(G)$ ($i = 1, 2$). Put $E_i = \{x; x \in G, f_i(x) \neq 0\}$. Then $\lambda(E_i) \neq 0$ so that $\lambda(E_i^{-1}) \neq 0$. The convolution of the characteristic functions of E_1 and E_2^{-1} is $\lambda(tE_1^{-1} \cap E_2^{-1})$ so that, using Fubini's theorem, $\int_G \lambda(tE_1^{-1} \cap E_2^{-1}) d\lambda(t) = \lambda(E_1)\lambda(E_2^{-1}) > 0$. Hence for some $x \in G$, $\lambda(x^{-1}E_1^{-1} \cap E_2^{-1}) > 0$, so that $\lambda(E_1x \cap E_2) > 0$. Now on the set $E_1x \cap E_2$ the functions $\delta_x * f_1, f_2$ are nonzero, so that the measures $\delta_x * \mu_1$ and μ_2 are not mutually singular (here δ_x denotes the measure obtained by placing unit positive mass at the point x).

THEOREM 2. Let μ be an absolutely continuous measure on G . Then

(i) for all $\nu \in M(G)$ with $\mu * \nu \neq 0$, the measures μ and $\mu * \nu$ are not of different type,

(ii) for all $\nu \in M(G)$ with $\nu \neq 0, \nu \neq \mu$, the measures ν and $\mu - \nu$ are not of different type.

Conversely if $\mu \in M(G)$ and satisfies (i) and (ii), then μ is absolutely continuous.

PROOF. Let μ be absolutely continuous. Then so is $\mu * \nu$ so that (i) holds (Lemma 2). For (ii), suppose ν and $\mu - \nu$ were of different type. Then one, ν say, is singular (Lemma 2). However, this would

imply $\mu - \nu$ and ν were not mutually singular, and hence not of different type, unless $\nu = 0$.

Conversely suppose $\mu \in M(G)$ satisfies (ii). Then it is clearly absolutely continuous or singular. However if $\mu \in M(G)$ is singular then we can find an absolutely continuous measure ν with $\mu * \nu \neq 0$. Then, since $\mu * \nu$ is absolutely continuous, μ and $\mu * \nu$ would be of different type.

4. Isometric isomorphisms of group algebras.

THEOREM 3. *Let G_1 and G_2 be two locally compact groups. Let ϕ be an isometric isomorphism of $M(G_1)$ onto $M(G_2)$. Let $\mu \in M(G_1)$. Then $\phi(\mu)$ is absolutely continuous if and only if μ is.*

PROOF. Theorem 1 shows that, if $\mu, \nu \in M(G_1)$ then they are mutually singular if and only if $\phi(\mu)$ and $\phi(\nu)$ are. Wendel's Theorem 3 [2] shows that $\nu \in H_1$ if and only if $\phi(\nu) \in H_2$, and hence $\mu, \nu \in M(G_1)$ are of different type if and only if $\phi(\mu)$ and $\phi(\nu)$ are. Theorem 2 then completes the proof.

COROLLARY. *Let G_1 and G_2 be two locally compact groups. The algebras $M(G_1)$ and $M(G_2)$ are isometrically isomorphic if and only if G_1 and G_2 are isomorphic topological groups.*

It also follows easily from Wendel's work that any isometric isomorphism ϕ from $M(G_1)$ onto $M(G_2)$ is of the form

$$\phi(\mu)(E) = \int_{\psi(E)} \chi(t) d\mu(t), \quad \mu \in M(G_1), E \text{ a compact set in } G_2,$$

where ψ is an isomorphism of G_2 onto G_1 and χ is a continuous character on G_1 .

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