SIMILARITY TRANSFORMATIONS OF HYPERSURFACES

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It is the purpose of this note to extend C. S. Hsü's theorem [4] to n dimensions, $n \geqq 2$. In order to do that we make use of the computations of [3, pp. 89–90].

Notations. M_1, M_2 are n-dimensional closed orientable Riemannian manifolds of class C^3, imbedded in $n+1$ dimensional Euclidean space. Δ_2 is the second differential operator of Beltrami. Δ_1 is the operator defined by

$$\Delta_1\sigma = g^{ij}\frac{\partial\sigma}{\partial x^i}\frac{\partial\sigma}{\partial x^j}.$$

As usual, g_{ij} is the positive-definite metric tensor of M_1 and $(g^{ij}) = (g_{ij})^{-1}$, $R = g^{ij}R_{ij}$, where R_{ij} is the Ricci tensor. We also use repeated index for summation. We denote corresponding elements of M_2 by attaching accents.

LEMMA [5, *p.* 30]. *In a compact Riemannian manifold with positive-definite metric, if a function σ satisfies*

$$\Delta_2\sigma \geqq 0$$

everywhere in the manifold, then σ is a constant.

THEOREM. *Given M_1, M_2 with positive R, R', respectively, and a diffeomorphism h: $M_1 \rightarrow M_2$ which preserves RI. (I is the first fundamental form.) Then h is a similarity.*

PROOF. It is sufficient to show that h followed by a homothetic transformation is a rigid motion. We first show that I'/I is a constant. Let $I'/I = R/R' = e^{2\sigma}$. Then $g'_{ij} = e^{2\sigma}g_{ij}$. By [3, pp. 89–90], we have

$$(R + 2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma) = R'e^{2\sigma}.$$

Making use of the hypothesis, we obtain

$$\Delta_2\sigma = -\left\{(n-2)/2\right\}\Delta_1\sigma.$$

Since $\Delta_1\sigma \geqq 0$,

$$-\Delta_2\sigma = \Delta_2(-\sigma) = \left\{(n-2)/2\right\}\Delta_1\sigma \geqq 0.$$

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By the lemma, σ is a constant. Consequently, I'/I is also a constant. Now we divide the proof into two different cases.

Case 1. For $n=2$, the fact that h followed by a homothetic transformation of a proportionality constant $(R'/R)^{-1/2}$ is the desired rigid motion follows from Cohn-Vossen's theorem.

Case 2. For $n>2$, the above fact follows from a different argument (cf. [**1**, pp. 26–27]). It says that isometric hypersurfaces in Euclidean space of dimension greater than three are congruent or symmetric.

References

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