ON BOUNDED FUNCTIONS WITH ALMOST-PERIODIC DIFFERENCES

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Let G be a multiplicative group, and let f be a complex-valued function on G . The *left differences* of f are the functions $\Delta_h f$, $\Delta_h f(x) = f(hx) - f(x)$, ($h, x \in G$). The following proposition is known [1; 2, p. 286], and is easy to prove.

LEMMA 1. *If G_1 is any topological group, if f_1 is a complex-valued function on G_1 , and if all the left differences of f_1 are continuous, then either f_1 is continuous or else f_1 is unbounded on every nonempty open subset of G_1 .*

One of several equivalent definitions of right almost-periodicity is the following:

DEFINITION. A complex-valued function f on a group G is *right almost-periodic* if, to every positive ϵ , there corresponds a finite number of elements of G , say $s_1, \dots, s_k$, such that to every t in G we can associate an integer $i \leq k$ for which

$$|f(xt) - f(xs_i)| < \epsilon, \quad \text{for all } x \in G.$$

Doss proved the following theorem [3]:

THEOREM. *Let G be a multiplicative group, and let the left differences $\Delta_h f$ be right almost-periodic for every $h \in G$, where f is a given complex-valued function on G . If f is bounded, then f is a right almost-periodic function.*

The proof given by Doss was elementary, depending only on the definition of right almost-periodicity. On the other hand, the similarity of the statements of Lemma 1 and the theorem suggests that the latter may follow naturally from the former. This is the case. The crucial tool is the following well-known result [4, p. 168].

LEMMA 2. *If G is any topological group, there is a compact group M and a continuous homomorphism α of G onto a dense subgroup G_1 of M such that a (continuous) function f on G is right almost-periodic if and only if there is a continuous function g on M such that $f(x) = g(\alpha(x))$ for all x in G .*

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PROOF OF THE THEOREM. Consider the given group G as a discrete topological group, and let M , α , and G_1 be as in Lemma 2. Let H denote the kernel of α . For each $h \in G$, $\Delta_h f$ is right almost-periodic on G , and can therefore be considered as a function on G_1 . That is

$$\Delta_h f(kx) = \Delta_h f(x), \quad \text{for all } k \in H, x, h \in G.$$

Taking $x = 1$, we obtain

$$(1) \quad f(hk) - f(h) = f(k) - f(1) \quad (k \in H, h \in G).$$

Taking $h = 1, k, \dots, k^{n-1}$ in (1) and adding the left and right-hand sides, respectively, of the resulting equations, we obtain

$$(2) \quad f(k^n) - f(1) = n[f(k) - f(1)] \quad (k \in H).$$

Since f is bounded, it follows that $f(k) - f(1) = 0$. Thus (1) shows that a function f_1 can be defined on G_1 such that $f_1(\alpha(x)) = f(x)$ for all x in G . In view of Lemma 2, it suffices to prove that f_1 has an extension as a continuous function on all of M .

Trivially, the left differences of f_1 correspond to left differences of f so that, by Lemma 2, the left differences of f_1 have continuous extensions to all of M . In particular, they are uniformly continuous on G_1 . By Lemma 1, then, f_1 is continuous on G_1 . Now let $y \in M$ and $\epsilon > 0$ be given. There is an open neighborhood V of 1 in M such that

$$(3) \quad |f_1(y_2) - f_1(y_1)| < \epsilon, \quad \text{for all } y_1, y_2 \in V \cap G_1.$$

Let s be an element of G_1 such that $y \in sV$. Since $\Delta_s f_1$ is uniformly continuous on G_1 , there is a symmetric neighborhood W of 1 in M such that $yW \subset sV$ and

$$(4) \quad |\Delta_s f_1(z_2) - \Delta_s f_1(z_1)| < \epsilon \quad (z_1, z_2 \in G_1, z_1^{-1}z_2 \in W^2).$$

Let t_1, t_2 be any two points of $yW \cap G_1$. Then we have

$$\begin{aligned} & |f_1(t_2) - f_1(t_1)| \\ &= |f_1(ss^{-1}t_2) - f_1(ss^{-1}t_1)| \\ &= |\Delta_s f_1(s^{-1}t_2) - \Delta_s f_1(s^{-1}t_1) + f_1(s^{-1}t_2) - f_1(s^{-1}t_1)| < 2\epsilon, \end{aligned}$$

by (3) and (4). Hence f_1 has a continuous extension to all of M , and the theorem follows.

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RESTRICTIONS OF FOURIER-STIELTJES TRANSFORMS

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1. Let G be a locally compact group with dual group Γ , and let dx and dy be the Haar measures on G and Γ , respectively. For a function $f \in L^1(G)$, the group algebra of G , the Fourier transform of f is denoted by $\hat{f}$:

$$\hat{f}(y) = \int_G (x, y)f(x)dx;$$

and for a measure $\mu \in M(G)$, the algebra of bounded measures on G , the Fourier-Stieltjes transform of μ is denoted by $\hat{\mu}$:

$$\hat{\mu}(y) = \int_G (x, y)d\mu(x).$$

Here (x, y) denotes the value of the character $y \in \Gamma$ at the point $x \in G$. Let A denote the family of Fourier transforms of functions $f \in L^1(G)$. For $F \subset \Gamma$, $\hat{f}|F$ denotes the restriction of $\hat{f}$ to F and $A|F = \{\hat{f}|F: \hat{f} \in A\}$. A function φ on Γ is said to be a multiplier of A provided $\varphi A \subset A$. It is a theorem of Helson's [2, Theorem 1] that the multipliers of A are precisely the Fourier-Stieltjes transforms. We are going to show that the obvious analogue persists on closed subsets of Γ , i.e., the multipliers of $A|F$ are precisely the almost everywhere restrictions to F of Fourier-Stieltjes transforms.

THEOREM. *Suppose φ is a function on Γ and $F \subset \Gamma$ is closed. In order that $\varphi|F = \hat{\mu}|F$ almost everywhere for some $\mu \in M(G)$, it is necessary and sufficient that*

$$(H) \quad \varphi A|F \subset A|F.$$

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