A PROBLEM ON BOUNDED ANALYTIC FUNCTIONS

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The following problem was proposed by members of the Colloquium on Classical Function Theory at Cornell University, 1961 [1]. "Is there a bounded analytic function defined in $|z|<1$ such that

$$N(r, 1/f')/(-\log(1-r)) \to 1 \qquad \text{as } r \to 1? \tag{1}$$

Can such an example be constructed as a gap series

$$f(z) = \sum c_n z^{\lambda_n}, \qquad \sum |c_n| < \infty ?"$$

Professor W. K. Hayman pointed out to me that I have already [2, Theorem IV] constructed functions with properties similar to those required. I show here that the extreme behaviour (1) occurs, for example, for the bounded analytic function

$$f(z) = \sum_1^\infty n^{-2} z^{\lambda_n},$$

where the sequence of positive integers $\{\lambda_n\}$ satisfies, as $n \to \infty$,

$$\lambda_{n+1}/\lambda_n \to \infty, \tag{2}$$

$$\log \lambda_{n+1} \sim \log \lambda_n. \tag{3}$$

To see this, put

$$r_n = \exp(-1/\lambda_n). \tag{4}$$

Then for all n and all real θ,

$$|f'(r_n e^{i\theta})| \geqq r_n |f'(r_n e^{i\theta})| \geqq e^{-1} n^{-2} \lambda_n - \Sigma_1 - \Sigma_2,$$

where $\Sigma_1 = \sum_1^n k^{-2}\lambda_k$, $\Sigma_2 = \sum_{k=n+1}^\infty k^{-2}\lambda_k r_n^{\lambda_k}$. From (2) and (4) the estimates $\Sigma_1, \Sigma_2 = o(n^{-2}\lambda_n)$ follow by easy calculations. Since $\log n = o(\log \lambda_n)$ by (2), we have therefore

$$\log |f'(r_n e^{i\theta})| > (1 - o(1)) \log \lambda_n$$

uniformly for $|\theta| \leqq \pi$; and noting that $|f'(r_n e^{i\theta})| = O(1)(1-r_n)^{-1} = O(\lambda_n)$ by (4) and the fact that f is bounded, we get

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uniformly for $|\theta| \leqq \pi$. But

$$N(r, 1/f') = (2\pi)^{-1} \int_{-\pi}^{\pi} \log |f'(re^{i\theta})| \, d\theta + O(1)$$

and so

$$N(r_n, 1/f') \sim \log \lambda_n.$$

From this (1) follows, because N is an increasing function of r and because $\log \lambda_n \sim -\log(1-r_n) \sim -\log(1-r_{n+1})$ by (3) and (4).

We note that, in the presence of (2) or even of the weaker hypothesis

(5) $$\liminf \lambda_{n+1}/\lambda_n > 1,$$

the condition (3) is necessary in order that the bounded analytic function $f(z) = \sum c_n z^{\lambda_n}$ should satisfy (1). Indeed, suppose that $c_n = O(n^p)$ for some fixed p; this is true a fortiori if f is bounded. Suppose also that (5) holds, but that, in contradiction of (3), there exist a fixed $\alpha > 1$ and infinitely many n such that

(6) $$\lambda_{n+1} > \lambda_n^{\alpha}.$$

Put $\rho_n = \exp(-1/\lambda_n^{\beta})$ where $1 < \beta < \alpha$. Then for $|z| = \rho_n$,

$$|f'(z)| = O(1)\left(\sum_{k=1}^{n} k^p \lambda_k + \sum_{k=n+1}^{\infty} k^p \lambda_k \rho_n^{\lambda_k} \right).$$

Using (5) we can show that the first sum on the right is $O(\lambda_n^{1+\epsilon})$ for every $\epsilon > 0$, and that the second sum is bounded for all n for which (6) is true. It is easy to conclude that, for every $\epsilon > 0$ and all large enough n for which (6) is true,

$$N(\rho_n, 1/f') < -\beta^{-1}(1+\epsilon) \log (1-\rho_n),$$

and it follows that (1) is false.

References

1. *Classical function theory problems*, Bull. Amer. Math. Soc. **68** (1962), 21; Problem 5.

2. P. B. Kennedy, *A property of bounded regular functions*, Proc. Roy. Irish Acad. Sect. A **60** (1959), no. 2, 7–14.

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