PATHOLOGICAL TOPOLOGIES AND RANDOM WALKS ON ABELIAN GROUPS

R. M. DUDLEY¹

It is shown in this paper that a part of the structure theory of locally compact abelian groups, which does not explicitly involve compactness, still fails to apply to complete separable metric abelian groups. Specifically, Theorem 2 asserts that any countable abelian group may be discretely imbedded in a metrizable abelian group with a dense cyclic subgroup. This result is independent of the rest of the paper.

Theorem 1 is a natural generalization of the result of [4], on the existence of recurrent random walks on discrete abelian groups, to locally compact abelian groups. Theorem 2 shows that it cannot be applied verbatim to complete separable metric groups. However, a statement in terms of dense subgroups rather than discrete ones is formulated as a conjecture.

We call a nonnegative finite measure μ defined on the Borel sets (the σ -field generated by the open sets) of a topological space *regular* if for every Borel set B and $\epsilon > 0$ there is a compact set $K \subset B$ such that $\mu(B) - \mu(K) < \epsilon$. It is known that any finite Borel measure on a complete separable metric space is regular, and that assuming the generalized continuum hypothesis "separable" is unnecessary (see [6]). Also, a bounded linear functional on the continuous functions on a compact Hausdorff space may be uniquely represented as the integral with respect to a regular Borel measure (see [5, §56]).

In order to define random walks on topological groups, we must define convolution of measures. Let G be a Hausdorff topological group and let μ and ν be finite regular Borel measures on G . Then there are disjoint compact subsets $C_1, C_2, \dots$, of G such that both μ and ν are concentrated in $\bigcup_{n=1}^{\infty} C_n$.

To obtain a product measure for μ and ν on $G \times G$, defined on all Borel sets, we may proceed as follows. On each compact set $C_m \times C_n$ we have a positive linear functional

$$f \rightarrow \int_{C_n} \int_{C_m} f(x, y) d\mu(x) d\nu(y)$$

Received by the editors August 21, 1962 and, in revised form, January 11, 1963.

¹ Some of the work on this paper was done while the author held a National Science Foundation Cooperative Fellowship in 1961-1962.

on the continuous functions on $C_m \times C_n$. (It follows from the Stone-Weierstrass theorem that the order of integration can be reversed.) This linear functional can, as mentioned, be written as

$$f \rightarrow \int f d\rho_{mn},$$

where ρ_{nm} is a regular Borel measure of total mass $\mu(C_m)\nu(C_n)$ on $C_m \times C_n$. The measure $\sum_{m,n=1}^{\infty} \rho_{mn}$ is then a regular Borel measure on $G \times G$ which is easily seen by mutual refinement to be independent of the choice of the sets C_n . We call this measure $\mu \otimes \nu$. It is easily seen to be an extension of the ordinary product measure $\mu \times \nu$.

If there is no countable base for the topology of G , it seems unclear whether $\mu \times \nu$ itself is defined on all Borel sets in $G \times G$, even after completion of the measure, a point not explained in [8]. Another way of obtaining a sufficiently widely defined product measure $\mu\nu$ is given in [1]. Since the $\mu\nu$ measure of any Borel set B is the supremum of the measures of closed sets included in B ,² and for any closed set F

$$\mu\nu(F) = \sum_{m,n=1}^{\infty} \mu\nu(F \cap (C_m \times C_n)),$$

$\mu\nu$ is regular. $\mu\nu$ has the property that

$$\int f d\mu\nu = \iint f(x, y) d\mu(x) d\nu(y)$$

for any Borel function f on $G \times G$. Applying this to functions equal to continuous functions on $C_m \times C_n$ and zero elsewhere, it follows that $\mu\nu$ and $\mu \otimes \nu$ agree on all Borel sets.

If ψ is a continuous mapping of a topological space A onto another space B and μ is a regular Borel measure on A , we can define a regular Borel measure μ_ψ on B by $\mu_\psi(M) = \mu(\psi^{-1}(M))$ for any Borel set M in B (see [8]). If g is a bounded Borel function on B and M is a Borel set in B , then

$$\int_M g d\mu_\psi = \int_{\psi^{-1}(M)} g(\psi(x)) d\mu(x)$$

since the equality holds for functions with finitely many values.

Now, if $p(x, y) = xy$ on $G \times G$, we define the convolution $\mu * \nu$ by

$$\mu * \nu = (\mu \otimes \nu)_p = (\mu\nu)_p.$$

² This property is stated in [1] only for B open, but since it also holds for B closed and for countable unions and intersections of sets where it holds, it is true of all Borel sets.

Then $\mu * \nu$ is a regular Borel measure on G satisfying, and characterized by,

$$\begin{aligned}\int f d(\mu * \nu) &= \int f(xy) d\mu\nu(x, y) = \int \int f(xy) d\mu(x) d\nu(y) \\ &= \int \int f(xy) d\nu(y) d\mu(x)\end{aligned}$$

for any bounded Borel function f . It follows directly that convolution is associative and is commutative if G is abelian. We also have

$$(\mu * \nu)(B) = \int \mu(By^{-1}) d\nu(y) = \int \nu(x^{-1}B) d\mu(x)$$

for any Borel set B , as in [8]. If the topology of G has a countable base, all these facts can be proved without recourse to [1] since we need only apply Fubini's theorem for the ordinary product measure.

Here is a fact relating convolution and the map $\mu \rightarrow \mu_\psi$:

LEMMA 1. *If G and H are topological groups, ψ is a continuous homomorphism of G into H , and μ and ν are regular Borel measures on G , then*

$$(\mu * \nu)_\psi = \mu_\psi * \nu_\psi.$$

PROOF. If B is a Borel set in H ,

$$\begin{aligned}(\mu_\psi * \nu_\psi)(B) &= \int_H \mu_\psi(Bh^{-1}) d\nu_\psi(h) \\ &= \int_G \mu_\psi(B\psi(y)^{-1}) d\nu(y) \\ &= \int_G \mu(\psi^{-1}(B\psi(y^{-1}))) d\nu(y) \\ &= \int_G \mu(\psi^{-1}(B)y^{-1}) d\nu(y) \\ &= (\mu * \nu)(\psi^{-1}(B)) = (\mu * \nu)_\psi(B), \quad \text{q.e.d.}\end{aligned}$$

To define a "random walk" on an abelian topological group G , we will have regular Borel probability measures μ and ν on G , where ν is an "initial distribution" and μ defines a "transition probability." Thus we want to define a probability measure Pr on the infinite product

$$G^\omega = G \times G \times G \cdots = G_0 \times G_1 \times G_2 \cdots$$

or space of sequences $\{x_n: n=0, 1, 2, \dots, x_n \in G\}$ such that the variables $x_0, x_1 - x_0, \dots, x_{n+1} - x_n, \dots$ are all independent and $\Pr(x_0 \in B) = \nu(B)$, $\Pr(x_{n+1} - x_n \in B) = \mu(B)$ for any $n \geq 0$ and Borel set $B \subset G$.

To do this, we first define a product measure $\nu \otimes \mu \otimes \mu \otimes \dots$ on G^ω . If $H_n = G_0 \times G_1 \times \dots \times G_n$, then we have a regular Borel measure

$$\nu \otimes \mu^{(n)} = \nu \otimes \mu \otimes \dots \otimes \mu$$

on H_n for each n (it is easy to check that the operation $\otimes$ is associative). For each n there is a natural mapping ψ_n of H_{n+1} onto H_n , and clearly

$$(\nu \otimes \mu^{(n+1)})_{\psi_n} = \nu \otimes \mu^{(n)}.$$

Thus all the hypotheses of a theorem of Bochner [2, Theorem 5.1.1, p. 120] on inverse limits of measures are satisfied. Hence there is a probability measure P on G^ω such that if ϕ_n is the natural mapping of G onto H_n for any n , $P_{\phi_n} = \nu \otimes \mu^{(n)}$. It is clear that for any closed set $F \subset G^\omega$ (for the product topology),

$$F = \bigcap_{n=1}^{\infty} \phi_n^{-1}(\phi_n(F)).$$

Thus P is defined on all Borel sets in G^ω . Since there are compact sets $\bigcap_{n=1}^{\infty} \phi_n^{-1}(K_n)$ in G^ω , where K_n is compact in H_n , with measure arbitrarily close to 1, P is regular on closed sets. It follows easily that P is also regular on open sets and hence on all Borel sets.

Finally, let RW be the mapping of G^ω into itself defined by

$$RW(y_0, y_1, y_2, \dots) = (y_0, y_0 + y_1, y_0 + y_1 + y_2, \dots).$$

RW is clearly continuous, so that $\Pr = P_{RW}$ is a regular Borel measure on G^ω . The measure space $(G^\omega, \Pr)$ will be called the random walk on G with initial distribution ν and transition probability μ . (If G has a countable base for its topology, P is the usual infinite product measure and our other definitions also coincide essentially with the usual ones.) A random walk will be called *recurrent* if for every open set $U \subset G$,

$$\Pr(x_n \in U \text{ for infinitely many } n) = 1.$$

THEOREM 1. *A locally compact abelian group G has a recurrent random walk if and only if (a) for every open subgroup H of G , G/H is countable, and (b) G does not have a discrete subgroup free abelian on three generators.*

To prove Theorem 1 we shall first generalize Lemma 1 of [2]. If (G^ω, Pr) is a recurrent random walk on G and U is an open subgroup of G , let t_0 be the least n such that $x_n \in U$, t_1 the next least, and so on; each t_j , $j=0, 1, 2, \dots$, is defined with probability one and equal to a Borel function on its domain. Let $x_n = x(n)$ and $y_n = x(t_n)$ for any n .

Now y_0 has a distribution ν_U and $y_1 - y_0$ has a distribution μ_U , where ν_U and μ_U are regular Borel measures on G . For given values of $t_0, \dots, t_n$, $t_{n+1} = t_n + k$ if and only if k is the least positive integer such that $x(t_n + k) - x(t_n) \in U$, since U is a subgroup. The set of variables $x(t_n + k) - x(t_n)$, $k > 0$, is independent of $x(t_0), \dots, x(t_n)$, by a standard argument. Thus $y_{n+1} - y_n$ is independent of $y_0, \dots, y_n$ and has distribution μ_U for all n . Thus if μ_n is the distribution of $(y_0, \dots, y_n)$ on H_n , the measures μ_n have an inverse limit Pr_U on G^ω , concentrated in U^ω , so that (U^ω, Pr_U) is a well-defined random walk on U with initial distribution ν_U and transition probability μ_U . The map $\{x_n\} \rightarrow \{x(t_n)\}$ of G^ω into U^ω obviously carries Pr into Pr_U .

LEMMA 2. (U^ω, Pr_U) is recurrent.

PROOF. For any open set $V \subset U$, V is open in G , so that

$$\text{Pr}\{x_n \in V \text{ infinitely often}\} = 1,$$

and since if $x_n \in V$, $n = t_m$ for some m ,

$$\begin{aligned} \text{Pr}\{x(t_m) \in V \text{ infinitely often}\} \\ = \text{Pr}_U\{y_m \in V \text{ infinitely often}\} = 1, \quad \text{q.e.d.} \end{aligned}$$

Let us now prove Theorem 1, beginning with "only if." Suppose G has a recurrent random walk. G cannot have an open subgroup H with G/H uncountable since for each n at most countably many cosets of H can be reached with positive probability by time n . To prove (b), we use the fact that G has an open subgroup H of the form $C \oplus V$ where C is compact and V is a vector group (see [7, pp. 160–162]; the restriction to second-countable topologies is removed in the second edition). Suppose J were a discrete subgroup of G , free on three generators. The sum of the ranks of G/H and $J \cap H$ is then at least 3, and the rank of $J \cap H$ is less than or equal to the dimension of V as a real vector space, since J is discrete. Let F be a free abelian subgroup of G/H of maximal rank, and let U be the union of all cosets of H belonging to F . Then U is an open subgroup of G of the form $C \oplus R^k \oplus Z^m$, where R and Z are the additive groups of the real numbers and the integers respectively and $k+m \geq 3$. (The relative

topology of U is the product topology indicated by the direct sum since G/H is discrete.) By Lemma 2, there exists a recurrent random walk on U . Applying the natural homomorphism ψ of U onto $R^k \oplus Z^m$ and using Lemma 1, we obtain a recurrent random walk on $R^k \oplus Z^m$, $k+m \geq 3$, which is impossible (see [3, p. 31]).

To prove "if," again let $H = C \oplus V$ be an open subgroup of G with C compact, V a vector group, and G/H countable. Since G/H is discrete, (b) implies that the sum of the rank of G/H and the dimension of V is at most 2.

Thus G/C has a countable dense subgroup M of rank at most 2. According to [4], there is a random walk on M recurrent on M as a discrete group and hence also with its relative topology from G/C . Thus there is a recurrent random walk on G/C with its transition probability μ concentrated in countably many points. We may assume μ has no mass at 0, according to the proof in [4].

Let ρ be a normalized Haar measure on C and each of its cosets. Let ξ be the "product" of μ and ρ defined by

$$\xi(B) = \int_{G/C} \rho(B \cap x) d\mu(x)$$

for each Borel set B . (Since μ is purely atomic, any function is measurable for it.) Probably ξ is the transition probability of a recurrent random walk, but the proof seems to be easier if we modify it slightly, letting $\sigma = (\xi + \rho_0)/2$, where ρ_0 is ρ confined to C .

It is easy to see that if a transition measure α defines a recurrent random walk on a group K for some initial distribution, and δ is the unit mass at the identity of K , then $\lambda\alpha + (1-\lambda)\delta$ is also the transition probability for a recurrent random walk with the same initial distribution if $0 < \lambda \leq 1$. Thus $(\mu + \delta)/2$ defines a recurrent random walk on G/C for any initial distribution, recurrent on M as a discrete group.

Let U be any open subset of G , and let A be a coset of C such that $\rho(A \cap U) > 0$ and $A \in M$. Then a random walk on G with initial distribution concentrated at 0 and transition probability σ has probability 1 of being in A infinitely often since it induces a random walk on G/C with transition probability $(\mu + \delta)/2$. If $x_n \in A$, then $\Pr(x_{n+1} \in A \cap U) = \frac{1}{2}\rho(A \cap U) > 0$ since μ has no mass at 0. The events that $x_{n(m)+1} \in A \cap U$, where $n(m)$ is the m th integer n such that $x_n \in A$, are independent for different m . Thus

$\Pr(x_n \in U \text{ infinitely often})$

$$\geq \Pr(x_n \in A \cap U \text{ infinitely often}) = 1, \quad \text{q.e.d.}$$

The following result shows that "locally compact" cannot be replaced by "complete separable metric" in Theorem 1.

THEOREM 2. *Let G be any countable abelian group. Then there is a Hausdorff topology T on the group $A = Z \oplus G$ with a countable base for which it is a topological group, Z is dense, and G has discrete relative topology.*

PROOF. Let $\{g_n\}_{n=1}^{\infty}$ be an enumeration of G in which every element occurs infinitely many times and let u_n be the element $(2^{2^n}, g_n)$ of A , $n = 1, 2, \dots$. For each n , let U_n be the set of all elements of A which can be written as finite sums $\sum a_i u_i$ where the a_i are integers such that $|a_i| \leq 2^{i-n}$ for all i . (Thus $a_i = 0$ for $i < n$.)

Then we have $U_n = -U_n$ for all n and $U_{n+1} + U_{n+1} \subset U_n$. Thus the U_n form a neighborhood-base at 0 for a topology $\mathfrak{J}$ on A making it a topological group (see [7, p. 55]). If $u \in U_n$ for some n , then $u = \sum a_i u_i$ with $a_i = 0$ for $i \geq k$ for some k . Thus $U_k + u \subset U_n$, so that the U_n are actually open sets (belong to $\mathfrak{J}$).

The sets $U_n + a$, $a \in A$, $n = 1, 2, \dots$, form a base for $\mathfrak{J}$. Hence to show that Z is dense in A for $\mathfrak{J}$ it suffices to show that for any n and $a = (z, g)$ there is an integer w such that $(w, 0) \in (z, g) + U_n$, or $(z - w, g) \in -U_n = U_n$. This is clear since $g = g_r$ for some $r \geq n$ and $u_r = (2^{2^r}, g_r) \in U_n$; let $w = z - 2^{2^r}$.

If $\sum_{j=1}^N a_j 2^{2^j} = 0$ with $|a_j| \leq 2^{j-1}$ for all j , a_j integers, then $a_j = 0$ for all j since

$$\left| \sum_{j=1}^n a_j 2^{2^j} \right| \leq n \cdot 2^{n-1} \cdot 2^{2^n} < 2^{2n+2^n} \leq 2^{2^{n+1}}$$

while

$$\left| \sum_{j=n+1}^N a_j 2^{2^j} \right| \geq 2^{2^{n+1}} \quad \text{if } a_{n+1} \neq 0.$$

Thus the only element of U_1 of the form $(0, g)$ is $(0, 0)$ so that the subgroup G of elements of this form has discrete relative topology. Finally if $u = (z, g) \neq (0, 0)$, then U_n and $U_n + u$ are disjoint if $0 < |z| < 2^{2^n}$ or if $z = 0$, $g \neq 0$, so that $\mathfrak{J}$ is Hausdorff. This completes the proof.

COROLLARY. *There is a topology for the free abelian group Z^4 on four generators for which it has a recurrent random walk, and in which there is a subgroup free on three generators with discrete relative topology.*

PROOF. Let $G = Z^3$ in Theorem 2 and recall that Z has a recurrent random walk.

To see that Theorem 1 fails for complete separable metric groups it suffices to take a completion of the group described in the corollary with its topology as given by Theorem 2. The uniformity of this topology is metrizable, and the completion has a natural structure of complete metric group with a dense cyclic subgroup and a discrete subgroup free on three generators.

One might conjecture, however, that a complete abelian topological group has a recurrent random walk if and only if it has a compact subgroup whose quotient has a dense subgroup of rank at most 2. "If" follows from the proof of Theorem 1.

I would like to thank the referee for suggesting a number of improvements and corrections in the proofs of both theorems.

REFERENCES

1. W. W. Bledsoe and A. P. Morse, *Product measures*, Trans. Amer. Math. Soc. **79** (1955), 173–215.
2. S. Bochner, *Harmonic analysis and the theory of probability*, Univ. of California Press, Berkeley, Calif., 1955.
3. K. L. Chung and W. H. J. Fuchs, *On the distribution of values of sums of random variables*, Mem. Amer. Math. Soc. No. 6 (1951), 12 pp.
4. R. M. Dudley, *Random walks on abelian groups*, Proc. Amer. Math. Soc. **13** (1962), 447–450.
5. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950.
6. E. Marczewski and P. Sikorski, *Measures in nonseparable metric spaces*, Colloq. Math. **1** (1948), 133–139.
7. L. S. Pontrjagin, *Topological groups*, Princeton Univ. Press, Princeton, N. J., 1946.
8. Karl Stromberg, *A note on the convolution of regular measures*, Math. Scand. **7** (1959), 347–352.

UNIVERSITY OF CALIFORNIA, BERKELEY