3. R. Doss, *On bounded functions with almost-periodic differences*, Proc. Amer. Math. Soc. 12 (1961), 488–489.

4. L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953.

OHIO STATE UNIVERSITY

RESTRICTIONS OF FOURIER-STIELTJES TRANSFORMS

JAMES WELLS

1. Let G be a locally compact group with dual group Γ, and let dx and dy be the Haar measures on G and Γ, respectively. For a function $f\in L^1(G)$, the group algebra of G, the Fourier transform of f is denoted by $\hat{f}$:

$$\hat{f}(y) = \int_G (x, y)f(x)dx;$$

and for a measure $\mu\in M(G)$, the algebra of bounded measures on G, the Fourier-Stieltjes transform of μ is denoted by $\hat{\mu}$:

$$\hat{\mu}(y) = \int_G (x, y)d\mu(x).$$

Here (x, y) denotes the value of the character $y\in\Gamma$ at the point $x\in G$. Let A denote the family of Fourier transforms of functions $f\in L^1(G)$. For $F\subset\Gamma$, $\hat{f}|F$ denotes the restriction of $\hat{f}$ to F and $A|F = \{\hat{f}|F: \hat{f}\in A\}$. A function φ on Γ is said to be a multiplier of A provided $\varphi A\subset A$. It is a theorem of Helson's [2, Theorem 1] that the multipliers of A are precisely the Fourier-Stieltjes transforms. We are going to show that the obvious analogue persists on closed subsets of Γ, i.e., the multipliers of $A|F$ are precisely the almost everywhere restrictions to F of Fourier-Stieltjes transforms.

THEOREM. *Suppose φ is a function on Γ and $F\subset\Gamma$ is closed. In order that $\varphi|F=\hat{\mu}|F$ almost everywhere for some $\mu\in M(G)$, it is necessary and sufficient that*

(H) $$\varphi A\,|\,F \subset A\,|\,F.$$

Presented to the Society, January 26, 1963, under the title *On function pairs related by a Fourier-Stieltjes transform*; received by the editors January 31, 1963.

If F is taken to be Γ we have the aforementioned result of Helson, while with F compact the result is trivial since there exists an $\hat{f}$ identically 1 on F. Of course this last assertion fails when F is noncompact. Thus our proof is actually designed for the case in which F is a proper, closed and noncompact subset of Γ.

2. Just as in [2], our proof ultimately rests on a theorem of Schoenberg [3]. However, the application we have in mind requires a mild alteration of the Schoenberg criterion. Following Eberlein [1] we say the function pair $\varphi, \eta \in L^\infty(\Gamma)$ satisfies the condition (S) provided there exists a constant M such that, for each $h \in L^1(\Gamma)$, the algebra of functions summable on Γ,

$$\text{(S)} \quad \left| \int_\Gamma h(y)\varphi(y)dy \right| \leqq M \sup_{x \in G} \left| \int_\Gamma (x, y)h(y)\eta(y)dy \right| = M \| (h\eta)^{\check{}} \|_\infty .$$

With η identically 1, condition (S) is the Schoenberg criterion. Given a function $h \in L^1(\Gamma)$ we denote by $h^{\check{}}$ the function on G defined by

$$h^{\check{}}(x) = \int_\Gamma (x, y)h(y)dy.$$

Lemma. *Condition* (S) *implies the existence of a* μ *in* $M(G)$ *such that* $\varphi = \eta\hat{\mu}$ *almost everywhere.*

Proof. The family of functions $(h\eta)^{\check{}}$ $(h \in L^1(\Gamma))$ is a subspace of $C_0(G)$, the sup norm Banach space of complex-valued continuous functions on G which vanish at infinity. We define (see [1, p. 466]) a functional ϕ on this subspace by

$$\phi((h\eta)^{\check{}}) = \int_\Gamma h(y)\varphi(y)dy \qquad (h \in L^1(\Gamma)).$$

According to condition (S) this definition yields a well-defined linear functional. Via the Hahn-Banach theorem and the Riesz representation theorem we conclude that ϕ has an extension ϕ^* to $C_0(G)$, $\|\phi^*\| = \|\phi\| \leqq M$, which is represented by a measure μ on G with $\|\mu\| \leqq M$. So, for each $h \in L^1(\Gamma)$,

$$\int_\Gamma h(y)\varphi(y)dy = \int_G (h\eta)^{\check{}}(x)d\mu(x) = \int_G \int_\Gamma (x, y)h(y)\eta(y)dyd\mu(x)$$
$$= \int_\Gamma h(y)\eta(y)\hat{\mu}(y)dy.$$

Hence $\varphi = \eta\hat{\mu}$ almost everywhere.

3. We are now in a position to establish the theorem. Only the sufficiency requires argument. First note that condition (H) implies φ is continuous relative to F. The set $kF=\{f: f\in L^1(G) \text{ and } \hat{f}(F)=0\}$, the kernel of F, is a closed ideal in $L^1(G)$ and we can form the quotient space $L^1(G)/kF$ in which the norm of each element $f+kF$, $f\in L^1(G)$, is the usual quotient norm $\|f+kF\|=\inf\{\|f+u\|_1: u\in kF\}$ under which $L^1(G)/kF$ becomes a Banach algebra.

In this quotient space setting condition (H) allows us to define a map U from $L^1(G)$ into $L^1(G)/kF$ by setting

$$U(f) = g + kF$$

if and only if

$$\hat{f}\varphi \mid F = \hat{g} \mid F.$$

Evidently U is linear. Let $f_n \to f$ in the norm of $L^1(G)$ and $U(f_n) = g_n + kF \to g + kF$ in the norm of $L^1(G)/kF$. According to the closed graph theorem, the continuity of U will follow if we can show that $U(f)=g+kF$. To this end observe that since $g_n - g + kF \to 0$ in $L^1(G)/kF$ there exists a sequence u_n in kF such that $g_n - g + u_n \to 0$ in $L^1(G)$. By hypothesis $\hat{f}_n\varphi \mid F = \hat{g}_n \mid F$, and since $\hat{u}_n$ vanishes on F, $\hat{f}_n\varphi \mid F = (\hat{g}_n + \hat{u}_n) \mid F$. Convergence in $L^1(G)$ implies uniform convergence in A so, letting n tend to infinity in this last expression, we have $\hat{f}\varphi \mid F = \hat{g} \mid F$ or $U(f)=g+kF$. Let M denote the norm of U.

Choose a net $\{e_\delta\}$ running through an approximate identity in $L^1(G)$ and set $U(e_\delta)=\varphi_\delta' + kF$. Fix c greater than 1. Since $\|\varphi_\delta' + kF\| \leq M\|e_\delta\|_1 = M$ we can choose φ_δ in $\varphi_\delta' + kF$ such that $\|\varphi_\delta\|_1 < cM$. By hypothesis the functions $\hat{e}_\delta$ converge to 1 uniformly on compact sets. This fact and the relation $\varphi\hat{e}_\delta \mid F = \hat{\varphi}_\delta \mid F$ imply that $\varphi = \lim \hat{\varphi}_\delta$ in the weak* topology of $L^\infty(F)$, i.e., if $h\in L^1(\Gamma)$ and η denotes the characteristic function of F,

$$\int_\Gamma h(y)\eta(y)\varphi(y)dy = \lim_\delta \int_\Gamma h(y)\eta(y)\hat{e}_\delta(y)\varphi(y)dy$$

$$= \lim_\delta \int_\Gamma h(y)\eta(y)\hat{\varphi}_\delta(y)dy = \lim_\delta \int_G (h\eta)^{\check{}}(x)\varphi_\delta(x)dx.$$

Hence

$$\left| \int_G h(y)\eta(y)\varphi(y)dy \right| \leq cM\|(h\eta)^{\check{}}\|_\infty.$$

Letting $c \to 1$ we have

$$\left| \int_G h(y)\eta(y)\varphi(y)dy \right| \leqq M\|(h\eta)^{\check{}}\|_\infty.$$

But this is just condition (S) with φ replaced by $\eta\varphi$ so, by the lemma, there exists a μ in $M(G)$ such that $\eta\varphi=\eta\hat{\mu}$ almost everywhere or $\varphi | F=\hat{\mu} | F$ almost everywhere. This completes the proof.

4. It is worthwhile to notice that this result applies to pairs of functions related by a Fourier-Stieltjes transform. Suppose we are given functions φ, η on Γ with the property $\varphi A \subset \eta A$. Then $(\varphi/\eta)A | F \subset A | F$, where F denotes the closure of $\{y: y\in\Gamma \text{ and } \eta(y)\neq 0\}$. By the theorem there exists a μ in $M(G)$ such that $(\varphi/\eta) | F=\hat{\mu} | F$; and since φ vanishes everywhere η does, the relation $\varphi=\eta\hat{\mu}$ holds almost everywhere.

References

1. W. F. Eberlein, *Characterizations of Fourier-Stieltjes transforms*, Duke Math. J. **22** (1955), 465–468.

2. H. Helson, *Isomorphisms of abelian group algebras*, Ark. Math. **2** (1953), 475–487.

3. I. J. Schoenberg, *A remark on the preceding note by Bochner*, Bull. Amer. Math. Soc. **40** (1934), 277–278.

University of Kentucky