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ON THE STRUCTURE OF THE GREEN'S OPERATOR

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1. Introduction. In the study of Cauchy problems of the form

$$du/dt + Au = f; \qquad u(\tau) = T \tag{1.1}$$

(where for example: $t \to u(t) \in \mathcal{E}^1(H)$ on $(\tau, b]$; $t \to u(t) \in \mathcal{E}^0(D(A))$ on $[\tau, b]$; H is a Hilbert space; $-A$ is a closed (unbounded) operator, infinitesimal generator of a strongly continuous semi-group; $\mathcal{E}^k(H)$ is the space of k-times continuously differentiable functions of t with values in H; the domain of A, $D(A)$, has the graph topology; and f, T are suitable), the solution takes the appearance

$$u(t) = G(t, \tau)u(\tau) + \int_\tau^t G(t, \xi)f(\xi)d\xi. \tag{1.2}$$

Formally the Green's operator $G(t, \xi)$ may be written $G(t, \xi) = \exp[-A(t-\xi)]$ (for general results in this direction see for example [**1**; **2**; **3**]). In this article we propose to study representations related to (1.2) for solutions of general operational differential equations $Su = f$ (the operators need not be differential operators of course but therein lies the motivation, see [**4**; **5**]; cf. also the papers [**3**; **6**; **7**; **8**; **9**; **10**]).

2. Basic framework. Let H be a Hilbert space and (S_0, S_0') a formally adjoint pair of closed densely defined operators in the sense

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Presented to the Society, April 19, 1963; received by the editors February 7, 1963.

[1] Research supported in part by the Rutgers Research Council.

of Browder [7]. Define $S_1 = S_0'^*$ (then $S_0 \subset S_1$) and let $H_0 = D(S_0)$, $H_1 = D(S_1)$, where H_0 and H_1 have the graph topology. Then $H_0 \subset H_0 \subset H$ (algebraically and topologically) and following [10] we set $H_1 = H_0 \oplus B$ where B is the so-called Cauchy space or space of abstract boundary conditions (see [7; 9; 10]). The symbol $\oplus$ denotes here an orthogonal direct sum (topological); when we wish to speak of a not necessarily orthogonal direct sum (topological) of two closed complementary subspaces M_1 and M_2 of a Hilbert space M we will write $M = M_1 + M_2$ (see here [11, p. 482]). It will be assumed throughout that S_0 is 1-1 with S_0^{-1} continuous and that S_1 is onto H. (Such hypotheses are verified in many problems of interest; they imply (see [7]) that S_0' has a closed range $R(S_0')$ and that (S_0, S_0') has a solvable realization operator $\tilde{S}$; $R(S_0)$ is clearly closed also.) Now we will call any topological supplement of H_0 in H_1 a Cauchy space Γ and write $H_1 = H_0 + \Gamma$ where in general H_0 and Γ are not orthogonal. Clearly any such Γ is isomorphic to B (both are isomorphic to H_1/H_0). Then operators $\hat{S}$ such that $S_0 \subset \hat{S} \subset S_1$ are characterized by linear subspaces $\hat{\Gamma}$ of Γ; that is, $\hat{H} = D(\hat{S})$ is the set $\{u_1 \colon u_1 \in H_1;\ ju_1 \in \hat{\Gamma} \subset \Gamma\}$ where $j \colon H_1 \to \Gamma$ is the (open) projection determined by H_0 and Γ. Then $\hat{H} = H_0 + \hat{\Gamma}$ and $\hat{H}$ would be given the graph topology. The following diagram will be useful in illustrating the subject (note $\ker S_1$ is closed in H or H_1)

$$
\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & 0 & \longrightarrow & \ker S_1 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_0 & \overset{i}{\longrightarrow} & H_1 & \overset{j}{\longrightarrow} & \Gamma \longrightarrow 0 \\
 & S_0 & \downarrow & & \downarrow & S_1 & \\
 & & R(S_0) & \overset{i}{\longrightarrow} & H & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0\,. & &
\end{array}
$$

The horizontal and vertical sequences are exact (and split by the Banach theorem of homomorphisms). The continuous maps i (injection), S_1, and j (projection) may be thought of as morphisms in the category of Hilbert spaces. Note now that $H_0 + \ker S_1$ is closed and hence a topological direct sum since if u_n is Cauchy in $H_0 + \ker S_1$ with $u_n = u_{0n} + u_{1n}$ then $S_0 u_{0n}$ converges which implies that u_{0n} converges. The diagram (2.1) may be further expanded as follows (cf. [7]), defining $\tilde{\Gamma}$ to be any topological supplement of $H_0 + \ker S_1$ in H_1.

$$
\begin{array}{ccc}
H_0 + \ker S_1 + \tilde{\Gamma} & \xrightarrow{j} & \{0\} + \Gamma_0 + \tilde{\Gamma} \\
\downarrow S_1 & & \\
R(S_0) + \{0\} + \tilde{H} & &
\end{array}
\tag{2.2}
$$

where $\Gamma_0 = j(\ker S_1)$, $\tilde{H} = S_1\tilde{\Gamma}$, and in abuse of notation we identify $\tilde{\Gamma}$ and $j\tilde{\Gamma}$. It is clear that $\tilde{H}$ is closed since S_1 is open and an isomorphism on $\tilde{\Gamma}$; to see that $\tilde{H} \cap R(S_0) = \{0\}$, suppose the contrary. Thus if $h_0 \in H_0$, $h \in \tilde{\Gamma}$, and $S_0 h_0 = S_1 h$, it follows that $h_0 - h \in \ker S_1$; since $h_0 - h \in H_1 - \ker S_1$ we must have $h_0 - h = 0$. Evidently $H = R(S_0) + \tilde{H}$.

We define the Green's operator to be the map $\mathcal{G}$: $(ju_1, S_1u_1) \to u_1$: $\Gamma \times H \to H_1$ which recovers u_1 from a knowledge of ju_1 and S_1u_1. It is seen from the diagrams that $\mathcal{G}$ is well defined (if $ju_1 = S_1u_1 = 0$ then $u_1 \in \ker S_1 \cap H_0 = \{0\}$). Moreover suppose $ju_1 \to 0$ in Γ and $S_1u_1 \to 0$ in H; then writing $u_1 = u_0 + u$, $u_0 \in H_0$, $u \in \Gamma$, we have $u \to 0$ in Γ and $S_1u + S_0u_0 \to 0$ in H. Hence $u \to 0$ in H, $S_1u \to 0$ in H, and $S_1u + S_0u_0 \to 0$ in H. This implies $S_0u_0 \to 0$ in H and therefore $u_0 \to 0$ in H by the continuity of S_0^{-1}. Thus finally $u_1 \to 0$ in H_1 and we have

PROPOSITION 1. *The map* $\mathcal{G}: \Gamma \times H \to H_1$ *defined by* $\mathcal{G}(ju_1, S_1u_1) = u_1$ *is continuous.*

It should be noted that $\mathcal{G}$ is not a bilinear map in the usual sense and is defined only on the set $G = \{(ju_1, S_1u_1)\} \subset \Gamma \times H$.

3. **Decomposition of the Green's operator.** By the preceding it follows that if $ju_1 = 0$ (i.e. $u_1 \in H_0$) then $\mathcal{G}(0, S_1u_1)$ defines a continuous map $\mathcal{G}_2$: $R(S_0) \to H_1$. Clearly on $R(S_0)$, $\mathcal{G}_2$ may be written as $S_0^{-1} = \tilde{S}^{-1}$ where $\tilde{S}$ is a solvable realization operator for (S_0, S_0'); hence $\mathcal{G}_2$ may be extended (as $\tilde{S}^{-1}$) to a continuous map $\mathcal{G}_2$: $H \to H_0 + \tilde{\Gamma}$ (cf. [7]). On the other hand if $u_1 \in \ker S_1$, then $\mathcal{G}(ju_1, 0)$ determines a continuous map $\mathcal{G}_1$: $\Gamma_0 \to H_1$ (the identity) which may be extended to a continuous map (the identity) $\mathcal{G}_1$: $\Gamma_0 + \tilde{\Gamma} \to H_1$. Then for $u_1 \in H_0 + \ker S_1$

$$
u_1 = \mathcal{G}_1(ju_1) + \mathcal{G}_2(S_1u_1), \tag{3.1}
$$

whereas for $u_1 \in \tilde{\Gamma}$ we must have

$$
u_1 = \mathcal{G}_1(ju_1) = \mathcal{G}_2(S_1u_1). \tag{3.2}
$$

Our interpretation of (1.2) is

$$
u_1 = \mathcal{G}_2(\rho S_1u_1) + \mathcal{G}_1(ju_1), \tag{3.3}
$$

where ρ: $H \to R(S_0)$ is the projection, determined by $R(S_0)$ and $\tilde{H}$. Another formula for the solution similar to (3.3) is

$$u_1 = \mathcal{G}_2(S_1u_1) + \mathcal{G}_1(\hat{\rho}ju_1), \tag{3.4}$$

where $\hat{\rho}: \Gamma \to \Gamma_0$ is the projection, determined by H_0 and ker S_1. Note that the split $H_0 + \ker S_1$ is predetermined; however there is still liberty in choosing $\tilde{\Gamma}$ and hence $\tilde{H}$.

We recall now the notion of a kernel for an operator $T: \mathcal{H} \to \mathcal{H}_1$ (see here for example [**12**; **13**; **14**]); we consider kernels in the sense of Aronszajn and will not attempt to treat here situations requiring the Schwartz kernel theorem (see [**15**]). Assuming $\mathcal{H}$ and $\mathcal{H}_1$ are separable Hilbert spaces of equivalence classes of measurable functions over a regular measure space (X, μ) (see [**12**]), then T has a kernel $T(y, \cdot)$ if: (1) for all $y \in X$, $T(y, \cdot) \in \mathcal{H}$; (2) the map $y \to T(y, \cdot): X \to \mathcal{H}$ is measurable; (3) for all $h \in D(T)$, $(Th)(y) = (h, T(y, \cdot))$ almost everywhere. If for example all functions in the range of a bounded operator T are continuous then following Theorem 4 of [**12**] it is seen that T has a kernel $T(y, \cdot)$. This will often prevail in applications (cf. [**17**]).

Suppose now that $\mathcal{G}_1$ and $\mathcal{G}_2$ have kernels $g_1(t, \cdot)$ and $g_2(t, \cdot)$; $\mathcal{G}_1$ and $\mathcal{G}_2$ are considered as operators in Γ and H respectively. Then for example (3.3) may be written (see [**16**] for extensions of (1.2))

$$u_1 = (\rho S_1 u_1, g_2(t, \cdot))_H + (ju_1, g_1(t, \cdot))_\Gamma. \tag{3.5}$$

We denote the adjoints of continuous maps T by ${}^tT$ and those of unbounded maps T by T^*. Then from (3.5), since $g_1 \in \Gamma$

$$u_1 = (u_1, {}^tS_1\, {}^t\rho g_2(t, \cdot) + {}^tjg_1(t, \cdot))_{H_1}. \tag{3.6}$$

The following exact sequences indicate how the maps work:

$$0 \to \tilde{H} \to H \xrightarrow{\rho} R(S_0) \to 0; \tag{1}$$

$$0 \to H \ominus R(S_0) \to H \xrightarrow{{}^t\rho} H \ominus \tilde{H} \to 0; \tag{2}$$

$$0 \to H_0 \to H_1 \xrightarrow{j} \Gamma \to 0; \tag{3}$$

$$0 \to H_1 \ominus \Gamma \to H_1 \xrightarrow{{}^tj} H_1 \ominus H_0 \to 0; \tag{4}$$

$$0 \to \ker S_1 \to H_1 \xrightarrow{S_1} H \to 0; \tag{5}$$

$$0 \to H \xrightarrow{{}^tS_1} H_1 \ominus \ker S_1 \to 0 \tag{6}$$

(note also ${}^tS_1: R(S_0) \to H_1 \ominus (\tilde{\Gamma} + \ker S_1)$ and ${}^tS_1: \tilde{H} \to H_1 \ominus (H_0 + \ker S_1)$). It is seen that certain problems arise because of the fact that even if $H_1 = (H_0 + \ker S_1) \oplus \check{\Gamma}$ it is not true necessarily that $H = R(S_0) \oplus \check{H}$, where $\check{H} = S_1\check{\Gamma}$. For example if we choose $\check{H}$ first, orthogonal to

$R(S_0)$, and define $\check{\Gamma} = {}^tS_1\check{H}$, then $\check{\Gamma}$ is orthogonal to $H_0 + \ker S_1$; however then $S_1\check{\Gamma} \neq \check{H}$ in general.

PROPOSITION 2. *Assume* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *have kernels as above; then* H_1 *has a reproducing kernel given by*

$$h_1(t, \cdot) = {}^tS_1\, {}^t\rho g_2(t, \cdot) + {}^tjg_1(t, \cdot). \tag{3.7}$$

We may relate ${}^tS_1$ to our original operators as follows. Assume $v \in H$ and ${}^tS_1v = w$; then for all $u \in H_1$ we have $(S_1u, v)_H = (u, w)_{H_1}$. This means $(S_1u, v - S_1w)_H = (u, w)_H$. Therefore $v - S_1w \in D(S_1^*)$ and since $S_1^* = S_0'$ it follows that $w = S_0'(v - S_1w)$ (recall H_1 is dense in H). Thus w appears as a solution of the equation $(v - S_1w) = (S_0')^{-1}w$. We note that $g_1(t, \cdot)$ as defined is a reproducing kernel for Γ and thus for $u_1 \in \Gamma$ there results $u_1 = (u_1, h_1)_{H_1} = (u_1, g_1)_\Gamma$. In general g_1 is the component of h_1 in Γ when H_1 is written in the form $\Gamma \oplus (H_1 \ominus \Gamma)$. It should be observed that H_0 orthogonal to $\ker S_1$ in H_1 is impossible and this fact is closely connected with the development which we have given. A result similar to (3.7) can also be obtained using (3.4). By virtue of the above we may now write (3.7) in a form suitable for calculation.

$${}^t\rho g_2 = ((S_0')^{-1} + S_1)(h_1 - {}^tjg_1). \tag{3.8}$$

This formula will not however entirely determine g_2 in terms of h_1 and g_1; it defines g_2 up to a term in $H \ominus R(S_0)$. However, this is sufficient and we have

PROPOSITION 3. *The component of* g_2 *in* $R(S_0)$ *is determined by* (3.8) *if* h_1 *and* g_1 *are known. If therefore* $\tilde{H}$ *is chosen orthogonal to* $R(S_0)$ (*with* $\tilde{\Gamma} = \tilde{S}^{-1}\tilde{H}$), *then* $\mathcal{G}_2(\rho S_1u_1)$ *is fully determined by* (3.8).

On the other hand let h_1 be given; then g_1 is determined as the component of h_1 in Γ when H_1 is decomposed as $H_1 = \Gamma \oplus (H_1 \ominus \Gamma)$. Thus if J is the orthogonal projection $J: H_1 \to \Gamma$ then $g_1 = Jh_1$. Define then the element ${}^t\rho g_2 = {}^tS_1^{-1}(h_1 - {}^tjg_1)$. This is well-defined since if $h_1 = \tilde{h}_1 + g_1$, $\tilde{h}_1 \in H_1 \ominus \Gamma$, $g_1 \in \Gamma$, then ${}^tjh_1 = {}^tjg_1 = \tilde{g}_1 \in H_1 \ominus H_0$ and since t_j is a projection $h_1 - \tilde{g}_1 \in H_1 \ominus \Gamma$; thus $h_1 - {}^tjg_1 \in H_1 \ominus \ker S_1$ with ${}^tS_1^{-1}(h_1 - \tilde{g}_1)$ well defined. Now since $h_1 - \tilde{g}_1 \in H_1 \ominus \Gamma$ we have ${}^tS_1^{-1}(h_1 - {}^tjg_1) \in R(S_0)$ and thus ${}^t\rho g_2 \in R(S_0)$. Assuming now $H = R(S_0) \oplus \tilde{H}$ with $\tilde{\Gamma} = \tilde{S}^{-1}\tilde{H}$, it follows that ${}^t\rho g_2$ defines an element g_2 $(= {}^t\rho g_2)$ in $R(S_0)$ with

$$\begin{aligned} (\rho S_1u_1, g_2)_H &= (S_1u_1, {}^t\rho g_2) = (S_1u_1, {}^tS_1^{-1}(h_1 - {}^tjg_1)) \\ &= (u_1, h_1 - {}^tjg_1) = u_1 - (ju_1, g_1) = \mathcal{G}_2(\rho S_1u_1). \end{aligned} \tag{3.9}$$

Hence $\mathcal{G}_2$ has a kernel g_2 in $R(S_0)$ given by ${}^t\rho^{-1}\, {}^tS_1^{-1}(h_1 - {}^tjg_1)$.

PROPOSITION 4. *Assume H_1 has a reproducing kernel h_1 and $H=R(S_0)\oplus\tilde{H}$. Then $\mathcal{G}_2$ has a kernel in $R(S_0)$ determined by* (3.8).

Added in proof. The results of this paper are used in constructing abstract Green's operators in [16] for problems related to [5]. It is shown that $\hat{S}=\check{S}^*$ (notations of [5]) and formulas such as (3.8) and (1.2) can be studied in more detail.

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