THE SHARPNESS OF SARIO'S GENERALIZED PICARD THEOREM[1]

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We present here the example referred to in L. Sario [**1**; **2**] which establishes the sharpness of his extensions of Picard's theorem. The nonintegrated estimate for the number of Picard values appears in [**1**], the integrated estimate in [**2**]. We use the notation introduced in these two papers.

The Riemann surface W will be described as an n-sheeted ramified covering of the nonextended z-plane. Consider n copies of this plane slit along the rays $\{\text{Re } z<0, \text{ Im } z=2\pi ih\}_{h=0,\pm 1,\ldots}$. Among these sheets identify the edges of the slits which belong to the same value of h so that $2\pi ih$ is a branch point of multiplicity n. The resulting surface has the capacity function $p=n^{-1}\log|z|$. The meromorphic function

$$w = \left(\frac{e^{z+\pi i}}{e^{z+\pi i}+1}\right)^{1/n}$$

is admissible. It omits the origin and the n values $e^{2\pi ih/n}$. The poles are at $z=2\pi ih$. W_m is the set $\{|z|\leqq e^{2\pi nm}\}$ lifted to W. Let Δ be a small neighborhood of ∞ in the w-plane. Using Hurwitz' formula one obtains

$$\overset{+}{e}_m \sim (n-1)e^{2\pi nm}/\pi,$$

$$n_m(\Delta) \sim e^{2\pi nm}/\pi,$$

$$\epsilon = \limsup \frac{\overset{+}{e}}{S} \leqq \limsup \frac{\overset{+}{e}_m}{n_m(\Delta)} = n-1.$$

But $\epsilon+2$ is never less than the number P of Picard values which is $n+1$. Consequently $\epsilon=n-1$ and the nonintegrated form of the generalized Picard theorem is sharp for every $P\geqq 2$.

The computations for the integrated form give

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$$E(k) \sim (n-1)e^{2\pi nk}/2\pi^2 n,$$
$$N(k, \infty) \sim e^{2\pi nk}/2\pi^2 n,$$
$$\eta = \limsup E(k)/T(k) \leqq \lim E(k)/N(k) = n - 1.$$

It follows that the integrated estimate $P \leqq 2+\eta$ is also sharp for $P \geqq 2$.

References

1. L. Sario, *Islands and peninsulas on arbitrary Riemann surfaces*, Trans. Amer. Math. Soc. **106** (1963), 521–533.

2. ———, *Meromorphic functions and conformal metrics on Riemann surfaces*, Pacific J. Math. **12** (1962), 1079–1098.

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