A SIMPLE CONSTRUCTION OF ANALYTIC FUNCTIONS WITHOUT RADIAL LIMITS

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The purpose of this note is to give a simple construction of a class of functions, analytic in the unit disc, having radial limits nowhere. We first prove, as a lemma, an easy Tauberian theorem, and use it to establish our result.

Let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of positive integers satisfying

$$\frac{r_k^{n_{k+1}}}{1 - r_k} \leqq \frac{1}{2^k}, \qquad k = 1, 2, 3, \cdots, \tag{1}$$

where r_k is defined as the positive solution of $r_k^{n_k} = 1 - 2^{-k}$. Such a sequence can easily be constructed by induction. Clearly $\lim_{k\to\infty} r_k = 1$; and if $k \leqq j$, then $1 - r_j^{n_k} \leqq 1/2^j$.

LEMMA. *Suppose a_i is a bounded sequence of complex numbers satisfying*

$$\lim_{r\to 1-} \sum_{k=1}^{\infty} a_k r^{n_k} = s.$$

Then

$$\sum_{k=1}^{\infty} a_k = s.$$

PROOF. Let A be a bound for the a_k. Then

$$\left| \sum_{k=1}^{j} a_i - \sum_{k=1}^{\infty} a_k r_j^{n_k} \right| \leqq \left| \sum_{k=1}^{j} a_k (1 - r_j^{n_k}) \right| + \left| \sum_{k=j+1}^{\infty} a_k r_j^{n_k} \right|$$
$$\leqq Aj/2^j + A/2^j.$$

THEOREM 1. *Suppose $\{b_k\}$ is a bounded sequence of complex numbers not satisfying* $\lim_{k\to\infty} b_k = 0$. *Then, if n_k satisfies* (1), $f(z) = \sum_{k=1}^{\infty} b_k z^{n_k}$ *is analytic in the unit disc, and $f(z)$ has radial limits nowhere.*

PROOF. Clearly, $f(z)$ is analytic in $|z| < 1$. Suppose $f(z)$ has a radial limit at $z = e^{i\theta}$, i.e.,

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$$\lim_{r \to 1-} \sum_{j=1}^{\infty} b_j e^{in_j\theta} r^{n_j}$$

exists. By the Lemma,

$$\sum_{j=1}^{\infty} b_j e^{in_j\theta}$$

exists; hence $\lim_{j \to \infty} b_j = 0$, contradicting the hypothesis.

The following theorem was suggested by a question of M. H. Heins.

THEOREM 2. *Let $\phi(r)$ be a continuous, strictly positive function, defined for $0 \leqq r < 1$, satisfying $\lim_{r \to 1-} \phi(r) = \infty$; let $\{b_k\}$ be a bounded sequence of complex numbers not satisfying $\lim_{k \to \infty} b_k = 0$. Then, there exists an increasing sequence of integers $\{n_k\}$, such that $f(z) = \sum_{k=1}^{\infty} b_k z^{n_k}$ is analytic in the unit disc, has radial limits nowhere, and $\max_{|z|=r} |f(z)| \leqq \phi(r)$.*

PROOF. Choose the integer n_1 so large that

$$|b_1| r^{n_1} \leqq 2^{-1}\phi(r), \tag{2}$$

for $0 \leqq r < 1$. Now, inductively, given n_k, choose n_{k+1} so large that (1) is satisfied, and that

$$|b_{k+1}| r^{n_{k+1}} \leqq 2^{-k-1}\phi(r), \tag{3}$$

for $0 \leqq r < 1$. Then, by Theorem 1,

$$f(z) = \sum_{k=1}^{\infty} b_k z^{n_k}$$

is analytic in the unit disc and has radial limits nowhere. By (2) and (3),

$$\max_{|z|=r} |f(z)| \leqq \sum_{k=1}^{\infty} |b_k| r^{n_k} \leqq \sum_{k=1}^{\infty} 2^{-k}\phi(r) = \phi(r).$$

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