HARNACK'S INEQUALITY AND THEOREMS ON MATRIX SPACES[1]

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1. Introduction. Let D denote the set of points formed by the class of $n\times n$ square matrices $z=(z_{jk})$ $(1\leqq j,\ k\leqq n)$ of complex numbers for which the Hermitian matrices $I-zz^*$ are positive definite, where $z^*=\bar{z}'$ and I, z' and $\bar{z}$ denote the identity, transposed and complex conjugate matrices, respectively. For simplicity, we use $H>0$ and $H\geqq 0$ to mean that the Hermitian matrix H is positive definite and positive semi-definite, respectively. Thus, $D=\{z: I-zz^*>0\}$. The variable z may be considered as a point in Euclidean space E_{2n^2} of $2n^2$ real dimensions. By defining a neighborhood of a point z, for example, as the set of points w such that $|w_{jk}-z_{jk}|<\epsilon$ for arbitrary positive ϵ $(1\leqq j,\ k\leqq n)$, it can be shown that the set D forms a convex bounded domain in E_{2n^2}. Let B denote the set of all $n\times n$ unitary matrices u. Thus the set B has dimension n^2 and B is a proper subset of the boundary of D. The closure of D is $\overline{D}=\{z: I-zz^*\geqq 0\}$.

Here we consider the differential operator [**3**; **4**]

$$\Delta = \sum_{j,k=1}^{n} \sum_{p,q=1}^{n} \left(\delta_{jp} - \sum_{r=1}^{n} z_{pr}\bar{z}_{jr}\right)\left(\delta_{kq} - \sum_{s=1}^{n} z_{sq}\bar{z}_{sk}\right)\frac{\partial^2}{\partial\bar{z}_{jk}\partial z_{pq}}\,. \tag{1}$$

A real-valued function ϕ possessing continuous first and second derivatives is said to be harmonic on D if

$$\Delta\phi(z) = 0, \qquad z \in D. \tag{2}$$

Hua has defined ϕ to be harmonic on the closure $\overline{D}$ of D, if it has continuous first and second partial derivatives on D, is continuous on $\overline{D}$ and satisfies (2) on $\overline{D}-B$ in the sense explained in [**3**, p. 1054]. It has been proved by Mitchell [**4**, p. 413] that if f is a real continuous function defined on B then the Poisson integral exists and is given by

$$\int_B f(u)P(z, u)dV, \tag{3}$$

where dV is Euclidean volume element on B and $P(z, u)$ is the Poisson kernel

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$$P(z, u) = \frac{1}{V}\left(\frac{\det(I - zz^*)}{\det((I - zu^*)(I - uz^*))}\right)^n \tag{4}$$

and V is the Euclidean volume of B. Formula (3) gives a harmonic function of z on D [**4**]. Since the Dirichlet problem is unique [**3**, p. 1057] with respect to functions harmonic on $\overline{D}$ as defined by Hua, any function ϕ harmonic on $\overline{D}$ can be expressed by means of the Poisson integral (3), where $f(u) = \phi(z) | B$ is continuous on B. It is also proved [**4**, p. 411] that $P(z, u)$ is harmonic for z in D and u in B. Furthermore $P(z, u)$ is positive on D.

In this paper we obtain upper and lower bounds of

$$\det((I - zu^*)(I - uz^*))$$

for fixed z in D and u ranging over B using the method of Lagrange multipliers. Though the upper bound of the determinant can be obtained easily by the well-known Hadamard's determinant theorem [**1**, p. 161], Lagrange's method enables us to get both bounds simultaneously. From these bounds we obtain upper and lower bounds of the Poisson kernel on B. This leads to a generalization of Harnack's inequality for non-negative harmonic functions in the case of one complex variable to our case. The so-called Harnack's first and second theorems are also established. The Harnack's inequality for the solutions of uniformly elliptic differential equations of second order in a domain of dimension $n \geqq 2$ was obtained by J. Moser [**5**].

2. **Harnack's inequality.** In this section we find the absolute maximum and minimum of the Poisson kernel defined by (1.4) for fixed z in D and u ranging over B.

For $z \in D$ it is known that there exist two unitary matrices u_0 and v_0 such that $z = u_0 R v_0$, where R is a diagonal matrix $R = [r_k] = [r_1, r_2, \cdots, r_n]$, $1 > r_k \geqq 0$, $k = 1, 2, \cdots, n$. Hence

$$\det(I - zz^*) = \det(I - RR^*) = \prod_{k=1}^{n} (1 - r_k^2), \tag{1}$$

where $r_k = r_k(z)$. Also

$$\det((I - zu^*)(I - uz^*)) = \det((v - R)(v^* - R)),$$

where $v = u_0^* u v_0^*$ is also a unitary matrix which ranges over the set B. Thus we wish to find the maximum and the minimum of

$$\det((v - R)(v^* - R))$$

on B for fixed R. Let

$$f = A\bar{A}, \tag{2}$$

where

$$A = \det(v - R) = \det(v_{jk} - \delta_{jk}r_j) \tag{3}$$

and $v_{jk} = x_{jk} + iy_{jk}$. Thus f is a polynomial in x_{jk} and y_{jk} $(1 \leqq j,\ k \leqq n)$ and takes its absolute maximum and minimum on B because it is a continuous function of x_{jk} and y_{jk} and B is a compact set in E_{2n^2}. Since $v \in B$, the set of $n \times n$ unitary matrices, we have the following n^2 conditions:

$$\begin{aligned} f_{jk} &= \sum_{t=1}^{n} (x_{jt}x_{kt} + y_{jt}y_{kt}) - \delta_{jk} = 0 \qquad (1 \leqq j \leqq k \leqq n), \\ g_{jk} &= \sum_{t=1}^{n} (x_{jt}y_{kt} - x_{kt}y_{jt}) = 0 \qquad\qquad (1 \leqq j < k \leqq n). \end{aligned} \tag{4}$$

Here we can find the maximum and minimum of f on B by Lagrange's method because it may be shown that (2) and (4) satisfy the hypotheses of a theorem in [1, p. 153]. Now, in order to apply Lagrange's method we set

$$F = A\bar{A} + \sum_{j,k=1;j\leq k}^{n} \lambda_{jk}f_{jk} + \sum_{j,k=1;j<k}^{n} \mu_{jk}g_{jk}, \tag{5}$$

where λ's and μ's are real numbers. Here, for convenience of later computation, we again introduce the complex expressions

$$\phi_{jk} = \sum_{t=1}^{n} v_{jt}\bar{v}_{kt} - \delta_{jk} = 0 \qquad (1 \leqq j,\ k \leqq n), \tag{6}$$

where $\phi_{kj} = \bar{\phi}_{jk}$,

$$f_{jk} = \frac{1}{2}(\phi_{jk} + \phi_{kj}), \qquad g_{jk} = \frac{1}{2i}(\phi_{kj} - \phi_{jk}). \tag{7}$$

Substituting (7) into (5) we get, after some rearranging,

$$F = A\bar{A} + \sum_{j,k=1}^{n} \nu_{jk}\phi_{jk}, \tag{8}$$

where $\nu_{jk} = \frac{1}{2}(\lambda_{jk} + i\mu_{jk})$ for $j \neq k$, $\nu_{kj} = \bar{\nu}_{jk}$ and $\nu_{jj} = \lambda_{jj}$. We also introduce the differential operators

$$\frac{\partial}{\partial v_{jk}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{jk}} - i\frac{\partial}{\partial y_{jk}}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{v}_{jk}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{jk}} + i\frac{\partial}{\partial y_{jk}}\right),$$

noting that $\partial F/\partial x_{jk}=0=\partial F/\partial y_{jk}$ implies $\partial F/\partial v_{jk}=0=\partial F/\partial \bar{v}_{jk}$ and conversely.

Now differentiate F in (8) with respect to v_{jk} and $\bar{v}_{jk}$ instead of differentiating F in (5) with respect to x_{jk} and y_{jk}. From (8) and (6) we get

$$\frac{\partial F}{\partial v_{jt}} = A_{jt}\overline{A} + \sum_{k=1}^{n} \nu_{jk}\bar{v}_{kt}, \tag{9}$$

where A_{jt} is the algebraic complement (the cofactor) of the element $v_{jt}-\delta_{jt}r_j$ in the determinant A. By setting $\partial F/\partial v_{jt}=0$, multiplying the right-hand side of (9) by v_{st} and summing t from 1 to n, and from (3) and (6) we obtain

$$-\nu_{jk} = r_k A_{jk}\overline{A} + \delta_{jk}A\overline{A}. \tag{10}$$

Similarly, calculating $\partial F/\partial \bar{v}_{jk}$, setting it equal to zero, multiplying the right-hand side by $\bar{v}_{st}$ and summing t from 1 to n, we get

$$-\nu_{jk} = r_j\overline{A}_{kj}A + \delta_{jk}A\overline{A}. \tag{11}$$

Comparing (10) and (11) we obtain

$$r_k A_{jk}\overline{A} = r_j\overline{A}_{kj}A. \tag{12}$$

Next, substituting ν_{jk} of (11) into (9) and setting $\partial F/\partial v_{jt}=0$,

$$A_{jt}\overline{A} - r_j A\sum_{k=1}^{n}\bar{v}_{kt}\overline{A}_{kj} - \bar{v}_{jt}A\overline{A} = 0. \tag{13}$$

Here by using (3), (12) and (13) we obtain

$$A_{jk} = A(\bar{v}_{jk} + \delta_{jk}r_k)/(1 - r_k^2) \tag{14}$$

since $A\neq 0$ for $z\in D$ and $v\in B$ [4, p. 410] and $r_j\neq 1$. Thus

$$\begin{aligned} 1/A &= A^{-n}\det(A_{jk}) = A^{-n}\det(A(\bar{v}_{jk}+\delta_{jk}r_k)/(1-r_k^2)) \\ &= \det(\bar{v}_{jk}+\delta_{jk}r_k)\Big/\prod_{k=1}^{n}(1-r_k^2). \end{aligned}$$

By setting $B=\det(\bar{v}_{jk}+\delta_{jk}r_k)$, we have $AB=\prod_{k=1}^{n}(1-r_k^2)$. Here since r_k occurs to at most the first power for each k in A and B and by unique factorization, A contains one and only one factor of $(1+r_k)(1-r_k)$ for each $k=1, 2, \cdots, n$. Hence $A\overline{A}=\prod_{k=1}^{n}(\epsilon_k-r_k)^2$, where $\epsilon_k=+1$ or -1, are all the possible values of relative maximum and minimum of $A\overline{A}$. Thus from the existence of the absolute maximum and minimum we finally obtain

$$\prod_{k=1}^{n} (1 - r_k)^2 \leqq A\bar{A} \leqq \prod_{k=1}^{n} (1 + r_k)^2. \tag{15}$$

This, together with (1), gives the following theorem.

THEOREM 1. *Let $z \in D$. Then*

$$\prod_{k=1}^{n} \frac{1 - r_k}{1 + r_k} \leqq \frac{\det(I - zz^*)}{\det((I - zu^*)(I - uz^*))} \leqq \prod_{k=1}^{n} \frac{1 + r_k}{1 - r_k} \tag{16}$$

for all u in B, where $z = u_0 R v_0$, u_0 and v_0 unitary matrices and $R = [r_1, \cdots, r_n]$ is a diagonal matrix with $1 > r_k \geqq 0$, $k = 1, \cdots, n$.

Thus we obtain bounds for the Poisson kernel (1.4) as

$$\frac{1}{V} \prod_{k=1}^{n} \left(\frac{1 - r_k}{1 + r_k}\right)^n \leqq P(z, u) \leqq \frac{1}{V} \prod_{k=1}^{n} \left(\frac{1 + r_k}{1 - r_k}\right)^n. \tag{17}$$

If we apply (17) to (1.3) with the assumption $f(u) \geqq 0$ on B, then we obtain Harnack's inequality

$$\prod_{k=1}^{n} \left(\frac{1 - r_k}{1 + r_k}\right)^n \phi(0) \leqq \phi(z) \leqq \prod_{k=1}^{n} \left(\frac{1 + r_k}{1 - r_k}\right)^n \phi(0). \tag{18}$$

3. **Harnack's theorems.** Harnack's first theorem can be obtained immediately for the domain D as follows.

THEOREM 2. *Let $\{\phi_p\}_{p \geq 1}$ be a sequence of real-valued functions, harmonic on $\bar{D}$, which converges uniformly on B. Then the limit function is harmonic on D.*

PROOF. From the uniqueness of the Dirichlet problem each function $\phi_p(z)$ harmonic on $\bar{D}$ can be expressed by means of the Poisson integral (1.3), that is,

$$\phi_p(z) = \frac{1}{V} \int_B \phi_p(u) P(z, u) dV, \qquad p = 1, 2, \cdots.$$

Let $\lim_{p \to \infty} \phi_p(u) = \phi(u)$ on B. Since ϕ_p is continuous on B which is compact, ϕ is also continuous on B. Hence by uniform convergence of $\{\phi_p\}$ on B and boundedness of $P(z, u)$ for fixed $z \in D$ and all $u \in B$,

$$\begin{aligned} \lim_{p \to \infty} \phi_p(z) &= \lim_{p \to \infty} \frac{1}{V} \int_B \phi_p(u) P(z, u) dV \\ &= \frac{1}{V} \int_B \phi(u) P(z, u) dV. \end{aligned}$$

Therefore, $\lim_{p\to\infty} \phi_p(z)$ exists for $z \in D$ and is harmonic on D.

Define a subdomain D_r, $1 \geqq r > 0$, of D by $D_r = \{z: I - r^{-2}zz^* > 0\}$.

LEMMA. *Let $z = u[r_k]v$, where u and v are unitary matrices. Then $z \in \overline{D}_r$ if and only if $0 \leqq r_k \leqq r$, $k = 1, 2, \cdots, n$.*

The necessary part of the proof follows from the fact that $z/r \in \overline{D}$ for $z \in \overline{D}_r$. The sufficient part of the proof can be easily seen by noticing that $r^2 I - zz^* = u[r^2 - r_k^2]u^*$.

THEOREM 3. *If $\{\phi_p\}_{p \geq 1}$ is a monotone nondecreasing sequence of harmonic functions on $\overline{D}$, and if ϕ_p tends to a finite limit at $z = 0$, then ϕ_p tends to a limit uniformly on $\overline{D}_r$, $0 < r < 1$, and the limit function is harmonic on D.*

PROOF. From the hypothesis that the sequence $\{\phi_p\}$ converges at $z = 0$, Harnack's inequality and the lemma, it can be proved that $\{\phi_p\}$ converges uniformly on $\overline{D}_r$, $0 < r < 1$. Secondly, setting $\psi_p = \phi_p - \phi_1$ on $\overline{D}$ and applying (2.18) to $\psi_p(u)$ and a theorem on interchanging $\int$ and $\lim_{p\to\infty}$ in [2, p. 582], we obtain

$$\lim_{p\to\infty} \int_B \psi_p(u)dV = \int_B \lim_{p\to\infty} \psi_p(u)dV.$$

This implies

$$\lim_{p\to\infty} \int_B \phi_p(u)dV = \int_B \lim_{p\to\infty} \phi_p(u)dV = \int_B \phi(u)dV.$$

This, together with (2.17), gives

$$\lim_{p\to\infty} \int_B P(z, u)\phi_p(u)dV = \int_B P(z, u)\phi(u)dV.$$

Therefore, since $\int_B P(z, u)\phi_p(u)dV = \phi_p(z)$ and $\phi(z) = \lim_{p\to\infty} \phi_p(z)$ it follows that $\phi(z) = \int_B P(z, u)\phi(u)dV$ and ϕ is harmonic on D.

REMARKS. (i) By using the left-hand inequality of (2.18) it can be proved that if $\phi_p(0)$ tends to ∞ as p tends to ∞, then ϕ_p tends uniformly to ∞ on every $\overline{D}_r$, $1 > r > 0$. Also it is not necessary to restrict z to be 0. (ii) It may be shown from the lemma that for any compact set $K \subset D$ there exists r, $1 > r > 0$, such that $K \subset \overline{D}_r$.

With these remarks it follows that a general form of Harnack's second theorem on D is

THEOREM 4. *Let $\{\phi_p\}_{p \geq 1}$ be a monotone nondecreasing sequence of harmonic functions on $\overline{D}$. Then there are only two possibilities: either*

ϕ_p tends uniformly to ∞ on every compact subset of D, or ϕ_p tends to a harmonic limit function ϕ on D, uniformly on every compact subset of D.

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