# A REMARK ON THE NONNORMAL LOCUS OF AN ANALYTIC SPACE

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**Notation.** By a ring we mean a nonnull commutative ring with identity. The radical of an ideal $\mathfrak{a}$ in a ring $R$ is denoted by $\operatorname{rad}_R \mathfrak{a}$ or $\operatorname{rad} \mathfrak{a}$. A ring is said to be normal if it is integrally closed in its total quotient ring. The conductor of a ring $R$ is denoted by $\mathfrak{c}(R)$, i.e., $\mathfrak{c}(R)$ is the ideal in $R$ consisting of those elements $r$ in $R$ for which $rs \in R$ for every element $s$ in the integral closure of $R$ in its total quotient ring; note that $R$ is normal if and only if $\mathfrak{c}(R) = R$.

Let $M$ be a multiplicative set in a ring $R$, let $S$ be the quotient ring of $R$ with respect to $M$, let $f$ be the natural homomorphism of $R$ into $S$, and let $K$ and $L$ be the total quotient rings of $R$ and $S$ respectively.[2] Then for any nonzerodivisor $v$ in $R$ we have that $f(v)$ is a nonzerodivisor in $S$; hence there exists a unique homomorphism $g$ of $K$ into $L$ such that $g(r) = f(r)$ for every $r \in R$; $g$ is again called the natural homomorphism of $K$ into $L$. Note that $L$ coincides with the total quotient ring of $f(R)$ and hence every element in $L$ can be written in the form $f(u)/f(v)$ where $u$ and $v$ are in $R$ and $f(v)$ is a nonzerodivisor in $f(R)$. Recall that the kernel of $f$ consists of those elements $r$ in $R$ for which $rm = 0$ for some $m$ in $M$; since $K$ is the total quotient ring of $R$, it follows that the kernel of $g$ consists of those elements $r$ in $K$ for which $rm = 0$ for some $m$ in $M$.

Let $X$ be an analytic space (over any algebraically closed complete nondiscrete valued ground field).[3] For any $p \in X$ let $R(p, X)$ denote the ring of analytic function germs in $X$ at $p$.[4] For any $p \in X$ and any analytic set germ $Y$ in $X$ at $a$ let $\mathfrak{i}(p, Y, X)$ denote the ideal in $R(p, X)$ consisting of those elements $r$ in $R(p, X)$ for which $r(Y) = 0$; if $Y$ is nonempty and irreducible then let $R(p, Y, X)$ denote the quotient ring $R(p, X)$ with respect to the prime ideal $\mathfrak{i}(p, Y, X)$. For any $p \in X$ and $Z \subset X$ let $Z_p$ denote the germ of $Z$ in $X$ at $p$. Let $\mathbf{S}(X)$ (resp. $\boldsymbol{N}(X)$) denote the set of singular (resp. nonnormal) points

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[2] For properties of quotient rings see [5, §§ 9–11 of Chapter IV] or [2, §18 of Chapter III].

[3] For definitions see [2].

[4] $R(a, X)$ does not contain any nonzero nilpotent elements, i.e., we consider only "reduced" analytic spaces.

of $X$, i.e., the set of points $p$ in $X$ for which $R(p, X)$ is not regular (resp. not normal).

**Introduction.** Let $X$ be an analytic space, let $p \in X$, and let $Y$ be a nonempty irreducible analytic set germ in $X$ at $p$.

Previously we have given the following analytic analogue of Zariski's criterion for a simple subvariety in algebraic geometry.[5]

*Criterion for a simple subspace.* $Y \not\subset \boldsymbol{S}(X)_p$ *if and only if* $R(p, X)$ *is regular.*[6]

In the same vein, here we give the following

*Criterion for a normal subspace.* $Y \not\subset \boldsymbol{N}(X)_p$ *if and only if* $R(p, X)$ *is normal.*

In [**2**, (46.28)] we have shown that $\boldsymbol{N}(X)$ is an analytic set in $X$ and $\mathfrak{i}(p, \boldsymbol{N}(X), X) = \operatorname{rad} \mathfrak{c}(R(p, X))$.[7] Also in [**2**, (24.3)] we have shown that for any prime ideal $\mathfrak{q}$ in $R(p, X)$, the integral closure of $R(p, X)/\mathfrak{q}$ in its quotient field is a finite $(R(p, X)/\mathfrak{q})$-module. Therefore the above criterion follows from the following proposition by taking $R = R(p, X)$ and $\mathfrak{p} = \mathfrak{i}(p, Y, X)$.

PROPOSITION. *Let $R$ be a noetherian ring such that* $\operatorname{rad}_R\{0\} = \{0\}$ *and for every prime ideal $\mathfrak{q}$ of $\{0\}$ in $R$ the integral closure of $R/\mathfrak{q}$ in its quotient field is a finite $(R/\mathfrak{q})$-module. Let $\mathfrak{p}$ be any prime ideal in $R$. Then the quotient ring of $R$ with respect to $\mathfrak{p}$ is normal if and only if* $\mathfrak{c}(R) \not\subset \mathfrak{p}$.

The "only if" part of the above proposition is proved in [**2**, (19.21)] where it is deduced from the corresponding assertion for an integral domain given in [**5**, p. 269]. The "if" part follows from the following slightly more general result by taking $M$ to be the complement of $\mathfrak{p}$ in $R$.

LEMMA 1. *Let $R$ be a noetherian ring such that* $\operatorname{rad}_R\{0\} = \{0\}$. *Let $S$ be the quotient ring of $R$ with respect to a multiplicative set $M$ in $R$. If* $\mathfrak{c}(R) \cap M \neq \emptyset$ *then $S$ is normal.*

We shall deduce the above lemma from the following two lemmas.

LEMMA 2. *Let $R$ be a noetherian ring such that* $\operatorname{rad}_R\{0\} = \{0\}$. *Let $S$ be the quotient ring of $R$ with respect to a multiplicative set $M$ in $R$. Let $K$ and $L$ be the total quotient rings of $R$ and $S$ respectively, and let $g$ be the natural homomorphism of $K$ into $L$. Then given any nonzerodivisor $y$ in $g(R)$ there exists a nonzerodivisor $v$ in $R$ such that $y = g(v)$. Furthermore* $g(K) = L$.

[5] See [**4**].

[6] For the complex case see [**1**, §9], and for the general case see [**2**, (45.11)].

[7] In the complex case this was first proved by Oka; see [**3**].

LEMMA 3.[8] *Let $M$ be a multiplicative set in a ring $R$ and let $S$ be the quotient ring of $R$ with respect to $M$. Let $K$ and $L$ be the total quotient rings of $R$ and $S$ respectively, let $R'$ and $S'$ be the integral closures of $R$ and $S$ in $K$ and $L$ respectively, and let $g$ be the natural homomorphism of $K$ into $L$. Assume that $g(K)=L$. Then $g(R')=S'$.*

**Proof of the lemmas.** First let us deduce Lemma 1 from Lemmas 2 and 3. Let the notation be as in Lemma 3 and assume that $R$ is noetherian and $\mathrm{rad}_R\{0\}=\{0\}$. Then by Lemmas 2 and 3 we get that $g(R')=S'$. Now assume that furthermore $\mathfrak{c}(R)\cap M\neq\emptyset$. Fix $w\in\mathfrak{c}(R)\cap M$ and let $z=g(w)$. Since $w\in M$ we get that $z$ is a unit in $S$. Since $g(R')=S'$, given any $z'\in S'$ there exists $w'\in R'$ such that $z'=g(w')$; since $w\in\mathfrak{c}(R)$ we get that $ww'\in R$ and hence $zz'=g(ww')\in g(R)\subset S$; since $z$ is a unit in $S$ we conclude that $z'\in S$. Thus $S'=S$, i.e., $S$ is normal.

Now let us prove Lemma 2. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_c$ be the distinct prime ideals of $\{0\}$ in $R$ labelled so that $\mathfrak{p}_i\cap M=\emptyset$ for $i=1, \cdots, a$, and $\mathfrak{p}_i\cap M\neq\emptyset$ for $i=a+1, \cdots, c$. Then $R\cap g^{-1}(0)=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_a$. Therefore $g(\mathfrak{p}_1), \cdots, g(\mathfrak{p}_a)$ are exactly the distinct prime ideals of $\{0\}$ in $g(R)$ and hence $g(\mathfrak{p}_1)\cup\cdots\cup g(\mathfrak{p}_a)$ is the set of all zerodivisors in $g(R)$. Given any nonzerodivisor $y$ in $g(R)$, take $v'\in R$ such that $g(v')=y$. Then $v'\notin\mathfrak{p}_i$ for $i=1, \cdots, a$. Relabel $\mathfrak{p}_{a+1}, \cdots, \mathfrak{p}_c$ so that $v'\notin\mathfrak{p}_i$ for $i=a+1, \cdots, b$, and $v'\in\mathfrak{p}_i$ for $i=b+1, \cdots, c$. For each $i>b$ we have that $\mathfrak{p}_j\not\subset\mathfrak{p}_i$ for $j=1, \cdots, b$, and hence $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_b\not\subset\mathfrak{p}_i$. Thus $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_b\not\subset\mathfrak{p}_i$ for $i=b+1, \cdots, c$ and hence $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_b\not\subset\mathfrak{p}_{b+1}\cup\cdots\cup\mathfrak{p}_c$,[9] i.e., there exists $v''\in R$ such that $v''\in\mathfrak{p}_i$ for $i=1, \cdots, b$, and $v''\notin\mathfrak{p}_i$ for $i=b+1, \cdots, c$. Let $v=v'+v''$. Then $v\notin\mathfrak{p}_1\cup\cdots\cup\mathfrak{p}_c$ and hence $v$ is a nonzerodivisor in $R$. Since $v''\in\mathfrak{p}_i$ for $i=1, \cdots, a$, we also have $g(v)=g(v')=y$. This completes the proof of the first assertion. Given any element in $L$ we can write it as $g(u)/g(u'')$ with $u$ and $u''$ in $R$ such that $g(u'')$ is a nonzerodivisor in $g(R)$. By what we have just proved, there exists a nonzerodivisor $u'$ in $R$ such that $g(u')=u''$. Now $u/u'\in K$ and $g(u/u')=g(u)/g(u'')$. This shows that $g(K)=L$.

Finally let us prove Lemma 3. Now $R'$ is integral over $R$ and hence $g(R')$ is integral over $g(R)$. Since $g(R)\subset S$, we get that $g(R')$ is integral over $S$ and hence $g(R')\subset S'$. To show that $S'\subset g(R')$, let $x'\in S'$ be given. Then there exist elements $x_1, \cdots, x_d$ in $S$ such that

$$x'^d + x_1x'^{d-1} + \cdots + x_d = 0.$$

[8] Compare with [5, Example 2 on p. 261].

[9] See [5, Remark on p. 215].

We can write $x_i = g(r_i)/g(m)$ with $r_1, \cdots, r_d$ in $R$ and $m$ in $M$. By assumption $g(K) = L$ and hence there exists $t' \in K$ such that $g(t') = x'$. Upon multiplying the above equation of integral dependence by $g(m^d)$ we get $g(q) = 0$ where

$$q = (t'm)^d + (r_1)(t'm)^{d-1} + (r_2m)(t'm)^{d-2} + \cdots + (r_dm^{d-1}).$$

Since $g(q) = 0$, there exists $m' \in M$ such that $qm' = 0$. Let $t = t'mm'$. Then

$$t^d + s_1t^{d-1} + \cdots + s_d = qm'^d = 0,$$

where

$$s_i = r_im^{i-1}m'^i \in R \qquad \text{for } i = 1, \cdots, d.$$

Therefore $t$ is integral over $R$, i.e., $t \in R'$. Since $mm' \in M$, we get that $g(mm')$ is a unit in $g(R)$ and hence $g(mm')$ is a unit in $g(R')$. Now $x'g(mm') = g(t'mm') = g(t)$, and hence $x' \in g(R')$.

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