SOME HAUSDORFF MATRICES NOT OF TYPE M

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Let $A=(a_{nk})$ denote an infinite matrix. Then A is said to be of type M if the conditions

$$(1) \qquad \sum_{n=0}^{\infty} |\alpha_n| < \infty, \qquad \sum_{n=0}^{\infty} \alpha_n a_{nk} = 0 \qquad (k = 0, 1, 2, \cdots)$$

always imply $\alpha_n = 0$ $(n=0, 1, 2, \cdots)$.

Matrices of type M were first introduced by Mazur [3] and so-named by Hill [2]. In [2], Hill developed several sufficient conditions for a Hausdorff matrix to be of type M. He showed that there exists a regular Hausdorff matrix not of type M. The particular matrix used contained a zero on the main diagonal. He also posed the following question: Does there exist a regular Hausdorff matrix which has no zero elements on the main diagonal and which is not of type M? The purpose of this note is to answer the above question in the affirmative and establish several other related theorems.

A matrix $A=(a_{nk})$ is called *triangular* if $a_{nk}=0$ for all $k>n$, and is called a *triangle* if A is triangular and $a_{nn}\neq 0$ for each n. (Some authors use the word normal instead of triangle.) Throughout this paper all matrices and sequences contain real entries.

If we use the words finite sequence to describe a sequence containing only a finite number of nonzero terms, it is clear that a triangular matrix which is not a triangle cannot be of type M, since a finite sequence can be found satisfying (1). Also, if a matrix is a triangle, there can be no finite sequence as a solution of (1). Hill's example is a triangular matrix, not a triangle.

Let $\mu=\{\mu_k\}$ be a sequence, Δ a forward difference operator defined by $\Delta\mu_k=\mu_k-\mu_{k+1}$, $\Delta^n\mu_k=\Delta(\Delta^{n-1}\mu_k)$, $n, k=0, 1, 2, \cdots$. Then a Hausdorff matrix $H=(h_{nk})$ is written in the form $h_{nk}=C_{n,k}\,\Delta^{n-k}\mu_k$ for $k\leqq n$, and $h_{nk}=0$ for $k>n$. The sequence μ is called the generating sequence for the matrix H, and, for a regular matrix, we have the representation

$$\mu_n = \int_0^1 u^n dq(u) \qquad (n = 0, 1, 2, \cdots),$$

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where $q(u)$ is a function of bounded variation on $0 \leqq u \leqq 1$, $q(0+) = q(0) = 0$, $q(1) = 1$, and $q(u) = [q(u+0) + q(u-0)]/2$ for $0 < u < 1$. For other properties of Hausdorff matrices see [1, XI]. The function $q(u)$ is commonly referred to as the mass function for μ.

In [2] it has been shown that, for regular Hausdorff matrices, (1) can be written in the form

$$\int_0^1 g_k(u)dq(u) = 0 \qquad (k = 0, 1, 2, \cdots), \tag{2}$$

where

$$g_k(u) = \sum_{n=k}^{\infty} \alpha_n C_{n,k} u^k (1-u)^{n-k} \qquad (k = 0, 1, 2, \cdots). \tag{3}$$

Each of the functions $g_k(u)$ represents an absolutely and uniformly convergent series on $0 \leqq u \leqq 1$, and, for every k,

$$g_k(u) = (-1)^k u^k g_0^{(k)}(u)/k! \tag{4}$$

THEOREM 1. *Let*

$$\mu_n = \frac{b(n-a)}{(-a)(n+b)}, \qquad a > 0, b > 0 \quad (n = 0, 1, 2, \cdots).$$

Then the corresponding regular Hausdorff matrix is not of type M.

PROOF. If a is a positive integer, then H is not of type M as remarked above, since it has a zero on its diagonal.

Assume a is not a positive integer. We may write μ_n in the form

$$\mu_n = -\frac{b}{a} + \frac{(a+b)}{a}\left(\frac{b}{n+b}\right),$$

and the corresponding mass function is

$$q(u) = \begin{cases} 1, & u = 1, \\ (a+b)u^b/a, & 0 \leqq u < 1. \end{cases} \tag{5}$$

Choose $\alpha_0 = 1$, $\alpha_n = (-1)^n a(a-1) \cdots (a-n+1)/n!$, $n \geqq 1$. Then $\sum_{n=0}^{\infty} |\alpha_n| < \infty$, and from (3) $g_0(u) = \sum_{n=0}^{\infty} \alpha_n (1-u)^n = u^a$. Using (4), we obtain $g_k(u) = (-1)^k a(a-1) \cdots (a-k+1)u^a/k!$, and the $g_k(u)$ satisfy (2). Therefore H is not of type M.

The concept of type M is of value only for reversible matrices, in particular, triangles. In general, it is superseded by the concept *perfect*; namely, that convergent sequences lie densely in the set of sequences transformed into convergent sequences by the matrix. For reversible matrices perfect and type M are equivalent.

Since Hill's example is not reversible, it is reasonable to ask if it is perfect. The Hausdorff matrix in the example has moment generating sequence

$$\mu_n = \frac{6(n-1)^2}{(n+1)(n+2)(n+3)}, \qquad n = 0, 1, 2, \cdots,$$

with corresponding mass function $Q(u) = 16u^3 - 27u^2 + 12u$, $0 \leqq u \leqq 1$. Instead of the corresponding Hausdorff matrix H, consider the matrix K, which agrees with H except that $k_{11} = 1$. Then H and K have the same convergence domains. Since K is a triangle, K is perfect if and only if it is of type M. Furthermore, if one chooses $\alpha_0 = \alpha_1 = 0$, $\alpha_n = 1/n(n-1)$ for $n \geqq 2$, then equations (3) and (4) are applicable, and

$$g_0(u) = \sum_{n=2}^{\infty} \frac{(1-u)^n}{n(n-1)}.$$

Term by term differentiation leads us to

$$g_0''(u) = \sum_{n=2}^{\infty} (1-u)^{n-2} = \frac{1}{u}.$$

Noting that $g_0'(1) = g_0(1) = 0$, we obtain $g_0(u) = u(\log u - 1)$. Using (3), the conditions in (2) become

$$\int_0^1 u\, dQ = 0,$$

$$\int_0^1 u \log u\, dQ = 0,$$

and

$$\int_0^1 (u \log u - u) dQ = 0.$$

It is easy to show that the first two are satisfied. The third condition is automatically satisfied, since it is a linear combination of the first two.

Therefore K is not of type M, and H is not perfect.

If one examines the sequences of Theorem 1 for a an integer, then one notes that the corresponding matrices are not reversible. It remains to determine if each such matrix is perfect.

THEOREM 2. *Let*

$$\mu_n = \frac{b(n-r)}{(-r)(n+b)}, \qquad a > 0,\ r \text{ a positive integer } (n = 0, 1, 2, \cdots).$$

Then the corresponding regular Hausdorff matrix is not perfect.

PROOF. The technique will be the same as that of the preceding example; that is, let K be H with $k_{rr}=1$. It remains to show that K is not of type M. For all $k>r$, equations (3) and (4) are valid. We wish to determine a sequence α which will satisfy (2). Using (5), (2) becomes

$$0 = \int_0^1 g_k(u)dq = \frac{b}{r}\left[-\alpha_k + (r+b)\sum_{n=k}^{\infty} \alpha_n \frac{\Gamma(n+1)\Gamma(k+b)}{\Gamma(k+1)\Gamma(n+b+1)} \right],$$

for $k=r+1,\ r+2, \cdots$. Solving the above system for the α_k's we obtain the recursion formula

$$\alpha_{k+1} = \frac{(k-r)\alpha_k}{k+1},$$

which gives us

$$\alpha_{r+m+1} = \frac{m!\alpha_{r+1}}{(r+2)(r+3)\cdots(r+m+1)}, \qquad m = 1, 2, 3, \cdots.$$

Selecting $\alpha_{r+1}=1$, we can show that $\sum_{m=1}^{\infty} |\alpha_{r+m+1}| < \infty$. Substituting in (3) we obtain $g_{r+1}(u)=u^r$. Using (4) leads to

$$g_0^{(r+1)}(u) = \frac{(-1)^{r+1}(r+1)!}{u}, \tag{6}$$

and hence that

$$g_k(u) = \frac{u^r}{\binom{k}{m-1}}, \qquad k = r+1, r+2, \cdots.$$

Thus conditions (2) reduce to showing that

$$\int_0^1 u^r\, dq = 0,$$

a condition easily verified.

There now remains the problem of determining the values α_0 through α_r. Because H has been modified in row r, it is not possible to use (6) to obtain $g_0^{(r)}(u)$ and hence $g_0(u)$. However, one can use (1) to determine α_r. Continued use of (1) will determine the remaining values $\alpha_{r-1}, \cdots, \alpha_0$. Thus K is not of type M, and H is not perfect.

The following is an open question. Does there exist a regular Hausdorff matrix which is perfect, but not of type M?

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References

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