ON PROJECTIONS WITH NORM 1—AN EXAMPLE

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Let X be a Banach space. It is a well-known result that if for every Banach space $Y \supset X$ with dim $Y/X=1$ there is a projection with norm 1 from Y onto X, then the same holds for every $Y \supset X$ without any restriction on Y/X (see, for example, [3] and the references given there). In [2] we proved that if for every $Y \supset X$ and every $\epsilon > 0$ there is a projection with norm $\leqq 1+\epsilon$ from Y onto X then there is also a projection with norm 1 from every $Y \supset X$ onto X.

In view of these results the following questions naturally arise (cf. also [3, problem 6]):

1. Let $Z \supset X$ be Banach spaces. Suppose that for every Y with $Z \supset Y \supset X$ and dim $Y/X=1$ there is a projection with norm 1 from Y onto X. Does there exist a projection with norm 1 from Z onto X?

2. Let $Z \supset X$ be Banach spaces. Suppose that for every $\epsilon > 0$ there is a projection with norm $\leqq 1+\epsilon$ from Z onto X. Does there exist a projection with norm 1 from Z onto X?

The answer to both questions is negative. This can be shown by rather simple examples. The purpose of this note is to show that even if the assumptions of both 1 and 2 hold, there may be no projection with norm 1 from Z onto X. We shall prove the following.

THEOREM. *There exist Banach spaces* $Z \supset X$ *with* dim $Z/X=2$ *satisfying*:

(i) *There is no projection with norm* 1 *from* Z *onto* X.

(ii) *For every* $\epsilon > 0$ *there is a projection with norm* $\leqq 1+\epsilon$ *from* Z *onto* X.

(iii) *For every* Y *with* $Z \supset Y \supset X$ *and* dim $Y/X=1$ *there is a projection with norm* 1 *from* Y *onto* X.

Before constructing the spaces Z and X we introduce some notations. Let K be the compact metric space of all the ordinals $\leqq \omega^2$ in the order topology.[2] Let K_m, $m=1, 2, \cdots$, be the subsets of K defined by

$$K_m = \{\alpha;\ (m-1)\omega < \alpha \leqq m\omega\}. \tag{1}$$

Clearly $K - \{\omega^2\} = \bigcup_{m=1}^{\infty} K_m$.[3] Let N denote the set of positive inte-

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[2] ω denotes, as usual, the ordinal number of the well-ordered set of the integers.

[3] $\{\omega^2\}$ denotes the set consisting of the single point ω^2. We do not consider 0 here as an ordinal number.

gers. Let $h(\alpha)$ be the function on K defined by

$$(2) \quad h(\alpha) = \begin{cases} 1 & \text{if } \alpha = m\omega + 2j - 1,\ m = 0, 1, 2, \cdots, j = 1, 2, \cdots, \\ -1 & \text{otherwise.} \end{cases}$$

Further let f_n, $n \in N$, be a sequence of continuous functions on K defined by

$$(3) \qquad f_n(\alpha) = \begin{cases} -1 & \text{if } \alpha \in K_{2m-1},\ m = 1, 2, \cdots, n, \\ 1 & \text{otherwise.} \end{cases}$$

The functions f_n converge (pointwise) as $n \to \infty$ to the function f defined by

$$(4) \qquad f(\alpha) = \begin{cases} -1 & \text{if } \alpha \in K_{2m-1},\ m = 1, 2, \cdots, \\ 1 & \text{otherwise.} \end{cases}$$

We are now ready to define the spaces Z and X. All the spaces will be over the real field. Let V be the space of all the bounded real-valued functions on the (abstract) set $K \times N$, with the usual vector operations and the sup as norm. As X we take the subspace of V consisting of all the functions v satisfying $v(\alpha, n) = v(\alpha, 1)$ for every $\alpha \in K$ and $n \in N$, and $v(\alpha, 1) \in C(K)$.[4] The mapping T_0 from X onto $C(K)$ defined by

$$(5) \qquad T_0 x(\alpha) = x(\alpha, 1), \qquad x \in X,\ \alpha \in K,$$

is clearly an isometry. As Z we take the subspace of V spanned by X and the functions

$$(6) \qquad z_1(\alpha, n) = f_n(\alpha), \qquad \alpha \in K,\ n \in N, \text{ and}$$

$$(7) \qquad z_2(\alpha, n) = h(\alpha)/n, \qquad \alpha \in K,\ n \in N.$$

Before turning to the proof that (i)–(iii) hold we remark that there is a projection with norm $\leqq \lambda$ from Z [resp. from a subspace Y of Z containing X] onto X if and only if there is an extension of T_0 from Z [resp. Y] onto $C(K)$ with norm $\leqq \lambda$. Further, we recall that there is a biunique correspondence between operators T from a Banach space U into $C(K)$ and w^* continuous mappings F from K to U^*. This correspondence is given by the equation $Tu(\alpha) = F(\alpha)u$, $\alpha \in K$, $u \in U$. Moreover, $\|T\| = \sup_\alpha \|F(\alpha)\|$ (cf. [1, p. 492]). The function F_0 from K to X^* corresponding to the operator T_0 defined in (5) is given by

[4] $C(K)$ denotes the Banach space of all the continuous real-valued functions on K with the sup norm.

$$(8) \qquad F_0(\alpha)x = x(\alpha, 1) = x(\alpha, n), \qquad x \in X,\ n \in N,\ \alpha \in K.$$

Proof of (i). By the preceding remarks and the Hahn-Banach Theorem we have to show that there is no function F from K to V^* having the following properties:

$$(9) \qquad F(\alpha)_{|X} = F_0(\alpha),^5 \qquad \alpha \in K,$$

$$(10) \qquad \|F(\alpha)\| = 1, \qquad \alpha \in K,$$

$$(11) \qquad F(\alpha)z_i \in C(K), \qquad i = 1, 2.$$

Suppose there is such an F. By the well-known representation of V^* we may consider each $F(\alpha)$ as a finitely additive measure on $K \times N$. Let α be an isolated point of K. The characteristic function χ_α of the set $\{\alpha\} \times N$ belongs to X and $F_0(\alpha)\chi_\alpha = 1$. Hence by (9) and (10) $F(\alpha)$ is a positive measure with norm 1 vanishing outside $\{\alpha\} \times N$. Since $\lim_{n\to\infty} z(\alpha, n)$ exists for every $z \in Z$ and $\alpha \in K$ it follows that for isolated $\alpha \in K$ there are non-negative numbers $a_{\alpha,n}$, $n \in N$, and $a_{\alpha,\infty}$ satisfying

$$(12) \qquad \sum_{n=1}^{\infty} a_{\alpha,n} + a_{\alpha,\infty} = 1, \text{ and}$$

$$(13) \qquad F(\alpha)z = \sum_{n=1}^{\infty} a_{\alpha,n} z(\alpha, n) + a_{\alpha,\infty} \lim_{n\to\infty} z(\alpha, n) \qquad z \in Z.$$

By (7) we obtain that $F(\alpha)z_2 = h(\alpha)\sum_{n=1}^{\infty} a_{\alpha,n}/n$. By the definition of h and by (11) it follows that as α tends to $m\omega$ ($m = 1, 2, \cdots$), $\sum a_{\alpha,n}/n$ tends to 0, and since the $a_{\alpha,n}$ are non-negative we obtain

$$(14) \qquad \lim_{\alpha\to m\omega} a_{\alpha,n} = 0, \qquad n, m = 1, 2, \cdots.$$

For isolated $\alpha \in K_{2m}$, $m = 1, 2, \cdots$, we obtain by (3), (4), (6), (12) and (13) that $F(\alpha)z_1 = 1$. Hence by (11)

$$(15) \qquad F(\omega^2)z_1 = 1.$$

For isolated $\alpha \in K_{2m+1}$, $m = 1, 2, \cdots$, we obtain similarly that

$$F(\alpha)z_1 = 2(a_{\alpha,1} + a_{\alpha,2} + \cdots + a_{\alpha,m}) - 1,$$

and hence by (11) and (14), $F((2m+1)\omega)z_1 = -1$. But this contradicts (11) and (15).

Proof of (ii). Let T_n be the linear operator from Z to $C(K)$ defined by

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[5] $F(\alpha)_{|X}$ denotes the restriction of $F(\alpha)$ to X.

$$T_n(x+\lambda z_1+\mu z_2)=x(\alpha,\ n)+\lambda z_1(\alpha,\ n),\qquad x\in X,\ \lambda,\ \mu \text{ scalars.}$$

Clearly $T_{n|X}=T_0$ for every $n\in N$. We estimate the norm of T_n. There exists an $M<\infty$ such that $|\mu|<M\|x+\lambda z_1+\mu z_2\|$ for every x, λ and μ. Hence

$$\begin{aligned}\|T_n(x+\lambda z_1+\mu z_2)\| &\leqq \sup_\alpha |x(\alpha,n)+\lambda z_1(\alpha,n)+\mu z_2(\alpha,n)| + |\mu|/n\\ &\leqq (1+M/n)\|x+\lambda z_1+\mu z_2\|.\end{aligned}$$

This proves (ii).

PROOF OF (iii). If Y is the subspace of Z spanned by X and z_2 then the operator T from Y to $C(K)$ defined by $T(x+\lambda z_2)=T_0(x)$ is a norm preserving extension of T_0. Hence we may assume that Y is the subspace of Z spanned by X and $y=z_1+\mu z_2$. By reversing the argument used in the proof of (i) it follows that it is sufficient to show that for isolated $\alpha\in K$ there exist non-negative $a_{\alpha,n}$, $n\in N$, and $a_{\alpha,\infty}$ satisfying (12) such that the function

$$g(\alpha) = \sum_{n=1}^{\infty} a_{\alpha,n}f_n(\alpha) + \mu h(\alpha)\sum_{n=1}^{\infty} a_{\alpha,n}/n + a_{\alpha,\infty}f(\alpha)$$

has a continuous extension to the whole of K. We choose the $a_{\alpha,n}$ and $a_{\alpha,\infty}$ as follows. For $\alpha\in K_{2m}$, $m=1, 2, \cdots$ and for $\alpha\in K_{2m+1}$ with $2m\leqq|\mu|$ we take $a_{\alpha,\infty}=1$ and $a_{\alpha,n}=0$, $n\in N$. For $\alpha\in K_{2m+1}$ with $2m>|\mu|$ we take $a_{\alpha,m}=1$, and $a_{\alpha,n}=a_{\alpha,\infty}=0$, $n\neq m$ if $\operatorname{sgn}\mu h(\alpha)=-1$,[6] while if $\operatorname{sgn}\mu h(\alpha)=1$ we take $a_{\alpha,m}=(2m-|\mu|)/(2m+|\mu|)$, $a_{\alpha,\infty}=1-a_{\alpha,m}$ and $a_{\alpha,n}=0$, $n\neq m$. With this choice of the $a_{\alpha,n}$ and $a_{\alpha,\infty}$ we have, for isolated α,

$$g(\alpha) = \begin{cases} -1 & \text{if } \alpha\in K_{2m+1} \text{ with } 2m\leqq|\mu|,\\ 1 & \text{if } \alpha\in K_{2m},\ m=1,2,\cdots,\\ 1-|\mu|/m & \text{if } \alpha\in K_{2m+1} \text{ with } 2m>|\mu|.\end{cases}$$

This g has, clearly, a continuous extension to K, and this concludes the proof of assertion (iii).

REFERENCES

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[6] We define $\operatorname{sgn} t=1$ if $t\geqq 0$ and $\operatorname{sgn} t=-1$ if $t<0$