for the norm $\|(x_1, \cdots, x_m)\| = \max_u |x_1\gamma_1(u) + \cdots + x_m\gamma_m(u)|$ is of standard type if there exist distinct values $u_1, \cdots, u_m$, such that $\gamma_j(u)$ is maximum at $u = u_j$, and $\gamma_i(u_j) = 0$, $i \neq j$, for $j = 1, \cdots, m$.

An unsolved problem concerning separable Banach spaces M is to show whether or not every such space has a base, that is, a set of elements $\{\gamma_j\}$ of M such that each element X of M has a unique expansion $X = \sum_{i=1}^{\infty} x_i\gamma_i$, which converges in norm to X. (See [3, pp. 110–111, and p. 238].) In case M has a base, the projections $P_jx = x_j\gamma_j$, although uniformly bounded, may have bounds greater than or equal to 2. If the unit cell C for a separable Banach space M is of standard type, then M has a base $\{\gamma_j\}$ with the stronger property that the projections P_j are all of norm 1.

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UNIVERSITY OF MIAMI

ON UNIFORM CONNECTEDNESS

S. G. MRÓWKA AND W. J. PERVIN[1]

Connectedness of topological spaces can be defined in terms of continuous functions to a discrete space (every continuous function to a discrete space is constant) or to the space of real numbers (every continuous real function has the Darboux property; i.e., the range of the function is an interval). We will consider in this paper similar properties for uniform and proximity spaces obtained by replacing continuous functions by uniformly continuous or equicontinuous functions (a function is equicontinuous iff it takes near sets into near sets).

DEFINITION 1. A uniform space $(X, \mathfrak{U})$ is uniformly connected iff every uniformly continuous function on X to a discrete space is constant.

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DEFINITION 2. A proximity space (X, δ) is equiconnected iff every equicontinuous function on X to a discrete space is constant.

A uniform (proximity) space is discrete iff its topology is discrete. In the above two definitions the discrete space may be assumed, without changing the sense of the definitions, to be the two-point discrete space $D=\{0, 1\}$.

THEOREM 1. *Let $(X, \mathfrak{U})$ be a uniform space and δ be the induced proximity relation. The following conditions are equivalent*:

(1) *X is equiconnected*;

(2) *every real-valued equicontinuous function on X has a range whose closure is an interval*;

(3) *for every $A \subset X$, $\emptyset \neq A \neq X$, A is close to $X-A$*;

(4) *the Smirnov compactification X^* of X is connected*;

(5) *X is uniformly connected*;

(6) *every real-valued uniformly continuous function on X has a range whose closure is an interval*;

(7) *X is chain connected*; *i.e.*, *for every p, $q \in X$ and every $U \in \mathfrak{U}$, $(p, q) \in U^n$ for some n.* (*The equivalence of* (5) *and* (7) *was stated, without proof, by Lubkin* [2].)

PROOF. The pattern of the proof will be (1)⇒(2)⇒(6)⇒(5)⇒(7) ⇒(3)⇒(4)⇒(1).

[(1)⇒(2)]. Let f be a real-valued equicontinuous function on X and suppose that $\mathrm{Cl}(f[X])$ is not an interval. Then there is an interval (a, b) such that $(a, b) \cap f[X]=\emptyset$, while $(-\infty, a] \cap f[X] \neq \emptyset$ and $[b, +\infty) \cap f[X] \neq \emptyset$. Setting $\phi(t)=0$ for $t \leqq a$ and $\phi(t)=1$ for $t \geqq b$ we see that ϕ is an equicontinuous function on the set $(-\infty, a] \cup [b, +\infty)$ into the discrete space D, hence $\phi \circ f$ is also equicontinuous, but it is not constant.

[(2)⇒(6)]. This is obvious since every uniformly continuous function is equicontinuous.

[(6)⇒(5)]. This follows from the fact that a uniformly continuous function f to D may be considered as a real-valued uniformly continuous function and the closure of its range will not be an interval unless f is constant.

[(5)⇒(7)]. Suppose there is a pair p, q of points in X and a set $U \in \mathfrak{U}$ such that $(p, q) \in U^n$ for no n. Define $f(x)=0$ if $(p, x) \in U^n$ for some n and $f(x)=1$ otherwise. Then f is a uniformly continuous function on X into D (indeed, if $(x, y) \in U$, then $f(x)=f(y)$), but f is not constant.

[(7)⇒(3)]. Let $A \subset X$, $\emptyset \neq A \neq X$, and let U be an arbitrary member of $\mathfrak{U}$. Take $p_0 \in A$, $q_0 \in X-A$, then $(p_0, q_0) \in U^n$ for some n. Hence there is a chain $p_0, p_1, \cdots, p_n = q_0$ of points in X such that $(p_{i-1}, p_i) \in U$ for $i = 1, 2, \cdots, n$. Denote by p_k the first of the points p_i that belongs to $X - A$. Hence $p_{k-1} \in A$ and (p_{k-1}, p_k) belongs to $U \cap [A \times (X - A)]$ which is then nonempty. Since U was an arbitrary member of $\mathfrak{U}$, A is close to $X - A$.

$[(3) \Rightarrow (4)]$. If X^* is not connected, then it is the union of two nonempty, disjoint sets A and B which are both open and closed in X^*. Since they are both closed and open and X is dense in X^*, $A = \mathrm{Cl}(A \cap X)$ and $B = \mathrm{Cl}(B \cap X)$ (closures being taken in X^*). Therefore $A \cap X$ is not close to $B \cap X = X - (A \cap X)$ although $A \cap X$ is a nonempty proper subset of X.

$[(4) \Rightarrow (1)]$. Let f be an equicontinuous function on X to D. Considering f as a real-valued function, it may be extended to a real-valued continuous function f^* on X^*. Since D is closed in the reals, f^* maps X^* into D. X^* being connected implies that f^* must be constant and hence f is constant.

It therefore follows from the above theorem that uniform connectedness is equivalent to equiconnectedness. Neither of these forms of connectedness is equivalent to (topological) connectedness, as the example of the rational numbers (with the usual uniformity or with the usual proximity) shows. However, connectedness does imply uniform connectedness since every uniformly continuous function is continuous. Furthermore, if a space is uniformly connected (or equiconnected) in every uniformity (or proximity) which is compatible with the topology, then the space is connected. This follows from the fact that X is connected whenever βX is connected and part (4) of the above theorem applies.

In connection with parts (2) and (6), we note that one cannot replace the condition given on the range of f by the requirement that the range of f be an interval as the example of the rationals with $f(x) = x$ shows. It is natural to ask what would happen if one would require that all equicontinuous functions on a proximity space have the Darboux property. The answer is rather unexpected.

THEOREM 2. *Let* (X, δ) *be a Lindelöf proximity space. If every real-valued equicontinuous function on* X *has the Darboux property, then* X *is connected* (*in the topological sense*).

PROOF. Suppose X is not connected and let $X = A \mid B$ be a separation of X. Let us denote by X^* the Smirnov compactification of X associated with the proximity δ and let us set $Z = \overline{A} \cap \overline{B}$ where the closures are taken in X^*. By a theorem of Smirnov [**3**], there exists a real-valued continuous function f on X^* such that $f(p) = 0$ for p in

Z but $f(p) > 0$ for p in X. Let us define a real-valued function g on X^* by setting $g(p) = f(p)$ for $p \in \bar{A}$ and $g(p) = -f(p)$ for $p \in \bar{B}$. Clearly the range of $g|X$ is not an interval since $g|X$ is never 0 but does take on positive and negative values. Since f is continuous on $\bar{A}$, $-f$ is continuous on $\bar{B}$, and f and $-f$ agree (both vanish) on $\bar{A} \cap \bar{B}$, g is continuous on X^*. Hence the restriction $g|X$ of g to X is equicontinuous.

EXAMPLE. The hypothesis of the Lindelöf property is essential in the above theorem as the following example shows. Let X be the discrete union of two copies of the "long line" (see [1, p. 55]) with the proximity induced by the one-point compactification. The space X is not connected since the two "long lines" form an obvious separation. On the other hand, the range of every equicontinuous function on X must be an interval. This follows from the fact that every continuous real-valued function on the "long line" is eventually constant. Using the one-point compactification, we see that the two constants obtained from the two copies of the "long line" must be the same. The initial segments of these copies are connected and hence their images are intervals and so the total image of X must be an interval.

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PENNSYLVANIA STATE UNIVERSITY