UNIQUENESS OF THE OPEN CONE NEIGHBORHOOD

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1. **Introduction.** The space $A\times[0, \infty)$ with $A\times 0$ identified to a point v is called the open cone $OC(A)$ over A and the point v is called the vertex of the cone.

Let X be a space. A point $x\in X$ is said to have an *open cone neighborhood* U if there is a homeomorphism f of some $OC(A)$ onto the open set U of X with $f(v)=x$. Our first theorem is the following.

THEOREM 1. *Let U and V be any two open cone neighborhoods of a point x in a locally compact Hausdorff space X. Then there is a homeomorphism of V onto U which leaves a neighborhood of x pointwise fixed.*

As immediate corollaries, we obtain a result of Mazur and Rosen that the open star of a vertex of a triangulated manifold is an open cell and also a result of Kwun and Raymond that the open star of a vertex on the boundary of a triangulated manifold with boundary is a cell minus a boundary point.

Theorem 1 was discovered when we tried to prove the following:

THEOREM 2. *Let M be a compact manifold with boundary. If M' is a compact manifold with boundary such that* Int $M=$ Int M' *then* Bd $M\times E^1=$ Bd $M'\times E^1$. *Conversely, if B is a compact manifold such that* Bd $M\times E^1=B\times E^1$ *then there exists a compact manifold M' with boundary such that* Int $M'=$ Int M *and* Bd $M'=B$.

Unfortunately, we do not know if Bd $M\times E^1=$ Bd $M'\times E^1$ does not imply Bd $M=$ Bd M'.

Finally the method of the proof of Theorem 1 may be used to prove more general theorems. For example,

THEOREM 3. *With x and X as in Theorem* 1, *if $U^1\subset U^2\subset\cdots$ is a sequence of open cone neighborhoods of x then $U=\bigcup U^i$ is also an open cone neighborhood of x homeomorphic to each U^i.*

This generalizes [1].

2. **Proof of Theorem 1.** Let f: $OC(A)\rightarrow X$ and g: $OC(B)\rightarrow X$, be homeomorphisms defining U and V respectively as open cone neighborhoods of x. Local compactness of X implies compactness of A and

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B. Hence we may assume that $X=U=OC(A)$ and f is the identity. We express each point of $OC(A)$ by (a, t), $a \in A$, $t \geqq 0$ with the understanding that $A \times 0$ is identified to x. The set $A \times t$ is abbreviated to A_t. We do similarly for $OC(B)$ and denote by U_t and V_t respectively the compact sets $\bigcup_{t' \leqq t} A_{t'}$ and $\bigcup_{t' \leqq t} g(B_{t'})$. Observe that if $t>0$, there exist $t', t''>0$ such that $V_{t'} \subset U_t$ and $U_{t''} \subset V_t$.

1. We find five positive numbers $p<q<r$, $s<t$ such that $g(B_s)$ separates A_p from A_q and $g(B_t)$ separates A_q from A_r. This is done by a repeated use of the above observation.

2. There exists a homeomorphism h_0 of $A \times [p, q]$ onto $A \times [q, r]$ such that

$$h_0(a, p) = (a, q),$$
$$h_0(a, q) = (a, r),$$
$$h_0 g(b, s) = g(b, t).$$

In particular, there is a homeomorphism F of $B \times [1,2]$ into $A \times [p, r]$ such that

$$F(b, 1) = g(b, s) \quad \text{and} \quad F(b, 2) = h_0 g(b, s).$$

This can be easily seen as follows. Let U_s' and V_s' denote the sets $\bigcup_{s' \geqq s} A_{s'}$ and $\mathrm{Cl}(U - V_s)$. We denote by ϵ a sufficiently small positive number. There exists a homeomorphism k_1 of U onto itself such that $k_1 | U_{q-\epsilon} \cup U'_{r+\epsilon} = 1$ and $k_1(a, q) = (a, r)$. There exists a homeomorphism k_2 of U onto itself such that $k_2 | V_{s-\epsilon} \cup V'_{t+\epsilon} = 1$ and $k_2 g(b, s) = g(b, t)$. There exists a homeomorphism k_3 of U onto itself such that $k_3 | U_{p-\epsilon} \cup U'_{q+\epsilon} = 1$ and $k_3(a, p) = (a, q)$. Let h_0 be the appropriate restriction of $k_3 k_2 k_1$.

3. Choose positive numbers u_i, w_j such that

$$q = u_0 < r = u_1 < u_2 < \cdots,$$
$$s = w_0 < w_1 < w_2 < \cdots$$

and

$$\lim u_i = \lim w_i = +\infty.$$

4. Let H_0 be the identity map of V_{w_0}. This can be extended to a homeomorphism H_1 of V_{w_1} onto $U_{u_0} \cup g(B \times [s, t])$.

5. Find a homeomorphism h_1 of $A \times [p, u_1]$ onto $A \times [q, u_2]$ which is an extension of h_0 such that $h_1(a, u_1) = (a, u_2)$.

Let H_2 be an extension of H_1 which maps V_{w_2} homeomorphically onto $U_{u_1} \cup h_1 g(B \times [s, t])$. ($H_2$ can be chosen so that $H_2(g(b, w_2)) = h_1 g(b, t)$, but this is not necessary.)

6. Find a homeomorphism h_2 of $A\times[u_0, u_2]$ onto $A\times[u_1, u_3]$ which is an extension of $h_1|A\times[u_0, u_1]$ such that $h_2(a, u_2)=(a, u_3)$.

Let H_3 be an extension of H_2 which maps V_{w_3} homeomorphically onto $U_{u_2}\cup h_2h_1g(B\times[s, t])$.

7. Similarly, find H_4, H_5, $\cdots$ which define a homeomorphism of V onto U which leaves V_{u_0} pointwise fixed.

3. Proof of Theorem 2.

The first part. Let M^* and M'^* be obtained from M and M' by shrinking Bd M and Bd M' to points p and p' respectively. By [2], p and p' have open cone neighborhoods homeomorphic to $OC(\text{Bd } M)$ and $OC(\text{Bd } M')$. Since M^* and M'^* are one-point compactifications of homeomorphic spaces Int M and Int M', there is a homeomorphism of M^* onto M'^* under which p is mapped into p'. By Theorem 1, $OC(\text{Bd } M)=OC(\text{Bd } M')$ with the vertices corresponding to each other. After deleting the vertices, Bd $M\times E^1=$ Bd $M'\times E^1$.

The second part. It follows that $OC(\text{Bd } M)=OC(B)$ with the vertices of the cones corresponding. Hence, an examination of a set like $V_s\cap U_p'$, where V_s and U_p' are the sets defined in the proof of Theorem 1 and s, p are positive numbers as in 2 of the proof of Theorem 1, reveals the existence of a compact manifold Y with boundary such that Bd Y is the disjoint union of B and Bd M and $Y-B=$ Bd $M\times[0, 1)$ with $y\in$ Bd M corresponding to $y\times 0$ and $Y-$ Bd $M=B\times[0, 1)$ with the points of B similarly corresponding.

We let $M'=M\cup Y$ with $M\cap Y=$ Bd M. Then clearly, M' is a compact manifold with boundary B. That Int M' is homeomorphic to Int M follows from [2].

4. Proof of Theorem 3.

Let f^i: $OC(A^i)\rightarrow X$ be homeomorphisms defining U^i as open cone neighborhoods of x. As in the proof of Theorem 1, A_t^i denotes the subset $\{(a^i, t)|a^i\in A^i\}$ of $OC(A^i)$ and $U_t^i=\bigcup_{t'\leq t}f^i(A_{t'}^i)$.

We can find, one by one, positive numbers t_{ij}, $i, j=1, 2, \cdots$ such that

$$\bigcup_j U_{t_{ij}}^i = U^i \qquad \text{for each } i, \tag{1}$$

$$U_{t_{ij}}^i \subset U_{t_{i+1,j}}^{i+1} - f^{i+1}(A_{t_{i+1,j}}^{i+1}) \tag{2}$$

for each i and j, and

$$U_{t_{ij}}^i \subset U_{t_{i,j+1}}^i - f^i(A_{t_{i,j+1}}^i). \tag{3}$$

In what follows, $U_{t_{ij}}^i$, etc. will be denoted simply by $U(i, j)$, etc.

Clearly for any sequence $j_1<j_2< \cdots$ of positive integers j_i, $\bigcup U(i, j_i)=U$.

We will repeatedly use the method of the proof of Theorem 1.

Choose j_1. Let H_1 be the identity map of $U(1, j_1)$. Choose $j_2>j_1$. As in 2 and 5 of the proof of Theorem 1, we extend H_1 to a homeomorphism H_2 of $U(1, j_2)$ onto a compact set F_2 containing $U(2, j_2)$ and contained in U^2. The next step reveals the general step. Since $H_2(f^1(A(1, j_2)))$ has a product neighborhood in U^2, we extend H_2 to a homeomorphism H_2' of U^1 into U^2. Consider the open cone neighborhood V^1 defined by $H_2'f^1$: $OC(A^1)\to X$. We find an integer $j_3>j_2$ so that $U(3, j_3-1)$ contains F_2. We extend the homeomorphism H_2: $V(1, j_2)\to F_2$ to a homeomorphism H_3' of $V(1, j_3)$ onto a compact set F_3 containing $U(3, j_3)$ and contained in U^3. Using H_3', we define a homeomorphism H_3, an extension of H_2, of $U(1, j_3)$ onto F_3. In exactly the same manner, we find $H_3, H_4, \cdots$ and they together generate a homeomorphism, leaving $U(1, j_1)$ pointwise fixed, of U' onto $\bigcup_i F_i=U$.

5. **Remarks.** As we mentioned earlier, we do not know of any compact nonhomeomorphic manifolds B and B' such that $B\times E^1=B'\times E^1$. Although the nonexistence, if proved, would strengthen Theorem 2, one might feel that such B and B' exist. One possibility is that $L(7, 1)\times S^n\neq L(7, 2)\times S^n$ for some n. Since $L(7, 1)\times E^n=L(7, 2)\times E^n$ for $n>2$ by [3], $\operatorname{Int}(L(7, 1)\times I^n)=\operatorname{Int}(L(7, 2)\times I^n)$ for $n>2$. Hence, by Theorem 2, $L(7, 1)\times S^{n-1}\times E^1=L(7, 2)\times S^{n-1}\times E^1$. But the remaining question is whether $L(7, i)\times S^{n-1}$ are homeomorphic.

Also in Theorem 3, we need not assume U to be the monotone union. It suffices to assume that U_t^i, $i=1, 2, \cdots, t>0$, form a cofinal family in the collection of the compact subsets of U. Hence the proof of Theorem 3 implies a result due to Stallings [4].

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