COMMUTATORS IN SEMI-SIMPLE ALGEBRAIC GROUPS

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Introduction. In [2] S. Pasiencier and H.-C. Wang proved that every element in a complex semi-simple Lie group is a commutator. The purpose of this note is to show that their method can be applied to the case of semi-simple algebraic groups without any restriction on the characteristic of the ground field. We shall prove the following

THEOREM. *In a connected semi-simple algebraic group defined over an algebraically closed field, every element is a commutator.*

For an additive analogue of the above theorem for semi-simple Lie algebras, see [3], where the algebraic closedness of the ground field is not assumed.

1. **Notation and preliminary.** We shall use the terminology and results in [1]. Let G be a connected semi-simple algebraic group defined over an algebraically closed field K, T a maximal torus of G, B a Borel subgroup containing T, B^u the unipotent part of B, N the normalizer of T in G, $W=N/T$ the Weyl group, and $X=X(T)$ the character group of T.

The character group X is a free abelian group of finite rank n. The Weyl group W acts on X by

$$(w\chi)(t) = \chi(\omega^{-1}t\omega), \tag{1.1}$$

where ω is an element in N representing w, and $\chi\in X$, $t\in T$. Also, X is equipped with a positive definite metric such that $2(\chi, \alpha)/(\alpha, \alpha)$ is an integer for any $\chi\in X$ and any root α. We shall normalize the metric such that (α, α) is an even integer for any root α, so that (χ, α) is an integer for any $\chi\in X$ and any root α. For any root α, the Weyl reflection w_α: $X\to X$ defined by $w_\alpha(\chi)=\chi-2(\chi, \alpha)(\alpha, \alpha)^{-1}\alpha$ belongs to W, and W is generated by the w_α.

For each root $\alpha>0$, there is a homomorphism $\tau_\alpha: K\to B^u$ of the additive group of K into B^u. Any element in B can be written uniquely in the form $t\prod_\alpha \tau_\alpha(x_\alpha)$, where $t\in T$, $x_\alpha\in K$, and where the product runs over all roots $\alpha>0$ in the increasing order. For any $t\in T$ we have $t\tau_\alpha(x)t^{-1}=\tau_\alpha(\alpha(t)x)$, and the commutator $(\tau_\alpha(x), \tau_\beta(y))$ can be written as a product $\prod\tau_\gamma(z_\gamma)$ with $\gamma\geqq\alpha+\beta$. From this we have

(1.2) *Let $\alpha_1, \alpha_2, \cdots, \alpha_M$ be all the positive roots in increasing order.*

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For $1 \leqq m \leqq M$, *denote by* B_m^u *the group generated by the* $\tau_{\alpha_i}(x)$, *where* $\alpha_i \geqq \alpha_m$ *and* $x \in K$. *Then* B_m^u *is a normal subgroup of* B^u.

2. **Main lemmas.** An element $t \in T$ is said to be *regular* if $\alpha(t) \neq 1$ for all roots α. As in [2] our theorem follows from the following three lemmas.

(2.1) *Every element in* G *is conjugate to an element in* B (*cf.* [1, *Exposé* 6, p. 13]).

(2.2) *If* $t \in T$ *is regular, then for any* $s \in B^u$, t *and* ts *are conjugate in* B^u.

(2.3) *For any* $t \in T$ *there exists a regular* $t_0 \in T$ *and an element* $\omega \in N$ *such that* $\omega^{-1}t_0\omega = t_0t$.

We shall prove (2.2) as the special case $m=1$ of the following:

(2.2′) *Let* α_m *and* B_m^u *be as in* (1.2). *Let* $t \in T$ *be regular. Then for any given* $s \in B_m^u$ *there exists an element* $u \in B_m^u$ *such that* $utu^{-1} = ts$.

If $m > M$, then (2.2′) is trivially true. We shall prove (2.2′) by descending induction on m. Suppose $m \leqq M$ and that (2.2′) is proved for greater m. We can write $s = \tau_\alpha(a)s'$, where $\alpha = \alpha_m$, and $s' \in B_{m+1}^u$. From the regularity of t we have $\alpha(t) \neq 1$, hence there exists $b \in K$ such that $\alpha(t)^{-1}b = a + b$. By (1.2) the element $s'' = \tau_\alpha(-b)s'\tau_\alpha(b)$ is clearly in B_{m+1}^u. Hence by the assumption of induction there exists $u' \in B_{m+1}^u$ such that $u'tu'^{-1} = ts''$. Now it is easy to verify that $u = \tau_\alpha(b)u'$ satisfies $utu^{-1} = ts$. Thus (2.2′) is proved.

3. **Proof of (2.3).** In order to obtain clarity we shall state a theorem of Kostant used in [2, p. 910].

(3.1) *Let* $X^{\mathbf{Q}} = X \otimes \mathbf{Q}$ *be the vector space over the field* $\mathbf{Q}$ *of rational numbers derived from* X *by extending the coefficient domain. For* $w \in W$, *set* $X_1^{\mathbf{Q}} = \{x \in X^{\mathbf{Q}} \mid w(x) = x\}$ *and let* $X_2^{\mathbf{Q}}$ *be the orthogonal complement of* $X_1^{\mathbf{Q}}$ *in* $X^{\mathbf{Q}}$. *Then* $X^{\mathbf{Q}} = X_1^{\mathbf{Q}} \oplus X_2^{\mathbf{Q}}$, *and* $X_2^{\mathbf{Q}} = (w-1)X_2^{\mathbf{Q}} = (w^{-1}-1)X_2^{\mathbf{Q}}$. *The subspace* $X_2^{\mathbf{Q}}$ *has a basis* $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ *consisting of roots such that*

$$w = w_{\alpha_1}w_{\alpha_2} \cdots w_{\alpha_m}. \tag{3.1.1}$$

Conversely, if w *is given by* (3.1.1) *and if* $\alpha_1, \alpha_2, \cdots, \alpha_m$ *are linearly independent roots, then* $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ *is a basis of* $X_2^{\mathbf{Q}}$. *If* α *is a root contained in* $X_2^{\mathbf{Q}}$, *then*

$$\dim(w_\alpha w - 1)X^{\mathbf{Q}} < \dim(w - 1)X^{\mathbf{Q}}.$$

Now let P be the set of all $\alpha \in X^{\mathbf{Q}}$ such that (χ, α) is an integer for any $\chi \in X$. Clearly all the roots are in P. For any $\alpha \in P$ and $z \in K^*$, define $t(\alpha, z) \in T$ by $\chi(t(\alpha, z)) = z^{(\chi,\alpha)}$, and denote by $T(\alpha)$ the group of all $t(\alpha, z)$, where $z \in K^*$.

(3.2) *If* $\alpha \in P$ *is a linear combination of* $\alpha_1, \alpha_2, \cdots, \alpha_m$ *in* P *with coefficients in* $\boldsymbol{Q}$, *then* $T(\alpha) \subseteq T(\alpha_1)T(\alpha_2) \cdots T(\alpha_m)$.

For the proof, let $0 \neq a \in \boldsymbol{Z}$ such that $a\alpha = \sum a_i\alpha_i$ with $a_i \in \boldsymbol{Z}$. For any $z \in K^*$, find $x \in K^*$ such that $x^a = z$. Then

$$\chi(t(\alpha, z)) = z^{(\chi,\alpha)} = \prod_i x^{(\chi, a_i\alpha_i)} = \chi\left(\prod_i t(\alpha_i, x^{a_i})\right).$$

Hence $t(\alpha, z) = \prod_i t(\alpha_i, x^{a_i})$. This proves (3.2).

(3.3) *If* $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ *is a basis of* $X^{\boldsymbol{Q}}$ *lying in* P, *then*

$$T = T(\alpha_1)T(\alpha_2) \cdots T(\alpha_n).$$

For the proof, let $(\chi_1, \chi_2, \cdots, \chi_n)$ be a basis of X. Then the $n \times n$ matrix $((\chi_i, \alpha_j))$ is nonsingular. Since K is algebraically closed, there exist $z_1, z_2, \cdots, z_n \in K^*$ such that

$$\prod_j z_j^{(\chi_i, \alpha_j)} = \chi_i(t) \qquad (1 \leqq i \leqq n)$$

for any given $t \in T$. Then $t = \prod_j t(\alpha_j, z_j)$. Thus (3.3) is proved.

For any $w \in W$ and $t \in T$ define t^w by $t^w = \omega^{-1}t\omega$, where ω is an element in N representing w. Set $t^{w-1} = t^w t^{-1}$, and denote by T^{w-1} the group of all t^{w-1}, where $t \in T$.

(3.4) *If* w *is given by* (3.1.1) *with linearly independent roots* $\alpha_1, \alpha_2, \cdots, \alpha_m$, *then* $T^{w-1} = T(\alpha_1)T(\alpha_2) \cdots T(\alpha_m)$. *Any* $t \in T^{w-1}$ *can be written in the form* $t = t'^{w-1}$, *where* $t' \in T^{w-1}$.

For any root α, we have $(w^{-1}-1)\alpha = \sum n_i\alpha_i$ with $n_i \in \boldsymbol{Z}$. Hence for any $z \in K^*$ and $\chi \in X$, we have

$$\begin{aligned}\chi(t(\alpha, z)^{w-1}) &= z^{(w\chi - \chi, \alpha)} = z^{(\chi, (w^{-1}-1)\alpha)} \\ &= \prod_i z^{(\chi, n_i\alpha_i)} = \chi\left(\prod_i t(\alpha_i, z_i)\right),\end{aligned}$$

where $z_i = z^{n_i}$. Hence $t(\alpha, z)^{w-1} = \prod_i t(\alpha_i, z_i)$. Since this is true for all roots α and $z \in K^*$, we have, by (3.3), $T^{w-1} \subseteq T(\alpha_1)T(\alpha_2) \cdots T(\alpha_m)$. On the other hand, by (3.1) there exists $\alpha \in P$ such that $(w^{-1}-1)\alpha = n\alpha_i$ with some nonzero integer n. Then $t(\alpha, z)^{w-1} = t(\alpha_i, z^n)$. Since $z \in K^*$ and i are arbitrary, it follows that $T(\alpha_1)T(\alpha_2) \cdots T(\alpha_m) \subseteq T^{w-1}$. Actually, (3.1) implies that for the given α_i one can take α in $P \cap X_2^{\boldsymbol{Q}}$. Then by (3.2) $t(\alpha, z) \in T(\alpha_1)T(\alpha_2) \cdots T(\alpha_m) = T^{w-1}$. Thus the second part of (3.4) is also proved.

Now we shall prove (2.3). Let $t \in T$ be given. By (3.3) $t \in T(\alpha_1)T(\alpha_2) \cdots T(\alpha_m)$ for some roots $\alpha_1, \alpha_2, \cdots, \alpha_m$, since $X^{\boldsymbol{Q}}$ is spanned by the roots. Let m be the least integer for which this is possible. Then by (3.2) the roots $\alpha_1, \alpha_2, \cdots, \alpha_m$ are linearly independent. Set

$$w = w_{\alpha_1}w_{\alpha_2} \cdots w_{\alpha_m}.$$

Then by (3.4),

$$t = t'^{w-1} \quad \text{with} \quad t' \in T(\alpha_1)T(\alpha_2) \cdots T(\alpha_m). \tag{3.5}$$

We shall show that

$$\alpha(t') \neq 1 \tag{3.6}$$

for any root α contained in $(w-1)X^Q$. Suppose the contrary. Then by (1.1)

$$\begin{aligned} \chi(t'^{w_\alpha w-1}) &= (w_\alpha(w\chi) - \chi)t' \\ &= (w\chi - \chi - 2(\chi, \alpha)(\alpha, \alpha)^{-1}\alpha)t' \\ &= (w\chi - \chi)t' = \chi(t'^{w-1}) = \chi(t) \end{aligned}$$

for all $\chi \in X$. Hence $t = t'^{w'-1}$, where we put $w' = w_\alpha w$. By (3.1), $m' = \dim(w'-1)X^Q < \dim(w-1)X^Q = m$. Hence by (3.1) and (3.4), $T^{w'-1} = T(\beta_1)T(\beta_2) \cdots T(\beta_{m'})$ for some roots $\beta_1, \beta_2, \cdots, \beta_{m'}$. But $t = t'^{w'-1} \in T^{w'-1}$. This contradicts the minimality of m. Thus (3.6) is proved.

For the element w, let $X^Q = X_1^Q \oplus X_2^Q$ be the decomposition given in (3.1). We have $(w-1)X^Q = X_2^Q$. Let $\gamma_1, \gamma_2, \cdots, \gamma_k$ be all the roots which are not in X_2^Q, and let $\gamma_i = \gamma_i' + \gamma_i''$, where $\gamma_i' \in X_1^Q$, $\gamma_i'' \in X_2^Q$. We shall show that if $z_1, z_2, \cdots, z_k \in K^*$ are suitably chosen and if we set $t'' = \prod t(c\gamma_i', z_i)$ with a nonzero integer c such that $c\gamma_i' \in P$ for $1 \leq i \leq k$, then $t_0 = t't''$ is the desired element. It is clear that $t''^{w-1} = 1$. Hence by (3.5), $t_0^{w-1} = t$. Also for any root α in X_2^Q, $\alpha(t'') = 1$, hence by (3.6) $\alpha(t_0) \neq 1$. Now for the roots γ_i, $1 \leq i \leq k$, we have

$$\gamma_i(t_0) = \gamma_i(t') \prod_j z_j^{(\gamma_i, c\gamma_j')}.$$

Since γ_i is not in X_2^Q, for each γ_i there exists an index j such that $(\gamma_i, c\gamma_j) \neq 0$. Since K has infinitely many elements, one can take $z_1, z_2, \cdots, z_k \in K^*$ such that $\gamma_i(t_0) \neq 1$ for $1 \leq i \leq k$. We have $t = t_0^{w-1}$, or $\omega^{-1}t_0\omega = tt_0$. Thus (2.3) is proved.

References

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