5. **Proof of Example 1.4.** If u were continuous, there would exist a compact $K \subset X$ such that f sufficiently small on K implies

$$|[u(f)](y)| < 1 \tag{5.1}$$

for all $y \in Y$. However, for any such K there exists a $y_0 \in Y - p(K)$, and if we pick $f \in C(X, R)$ to be 0 on K and 2 on $p^{-1}(y_0)$, then $[u(f)](y_0) = 2$, and hence f fails to satisfy (5.1). This implies that u cannot be continuous.

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A SHORT PROOF OF THE ARENS-EELLS EMBEDDING THEOREM

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In this note, we give a short proof of the following theorem of R. Arens and J. Eells.

THEOREM. *Every metric space can be embedded isometrically as a closed, linearly independent subset of a normed linear space.*

PROOF. Observe first that it suffices if one can always find an isometric, linearly independent embedding. For if M is any metric space, and if its completion M^* is thus embedded in a normed linear space E, then M^* (being complete) is closed in E, so that M is closed

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in the subspace E_M of E which it spans algebraically (since $M = E_M \cap M^*$).

So let X be any metric space. Let Y be a metric space (with metric d) which contains X and one point $y_0 \notin X$. Let $\mathrm{Lip}(Y)$ denote the set of all real-valued functions f on Y such that $f(y_0) = 0$ and, for some $K \geqq 0$,

$$|f(x) - f(y)| \leqq K\mathrm{d}(x, y), \qquad x, y \in Y;$$

denote the smallest K which works for such an f by $\|f\|$. This makes $\mathrm{Lip}(Y)$ into a (complete) normed linear space. Let E be its dual, with the usual norm

$$\|\phi\| = \sup\{ |\phi(f)| : \|f\| \leqq 1\}.$$

Define $h: X \to E$ by $h(x) = \bar{x}$, where $\bar{x}(f) = f(x)$ for all $f \in \mathrm{Lip}(Y)$.[2]

To see that h is an isometry, let $x_1, x_2 \in X$. Then $\|\bar{x}_1 - \bar{x}_2\| \leqq \mathrm{d}(x_1, x_2)$ from the definitions. On the other hand, if $g(y) = \mathrm{d}(y, x_2) - \mathrm{d}(y_0, x_2)$ for all $y \in Y$, then $g \in \mathrm{Lip}(Y)$, $\|g\| = 1$, and $(\bar{x}_1 - \bar{x}_2)(g) = \mathrm{d}(x_1, x_2)$, so that $\|\bar{x}_1 - \bar{x}_2\| \geqq \mathrm{d}(x_1, x_2)$.

To see that $h(X)$ is linearly independent in E, let $x_1, \cdots, x_{n+1}$ be distinct elements of X. Then $\bar{x}_{n+1}$ cannot be a linear combination of $\bar{x}_1, \cdots, \bar{x}_n$, for if $g(y) = \mathrm{d}(y, \{y_0, x_1, \cdots, x_n\})$ for $y \in Y$, then $g \in \mathrm{Lip}(Y)$, and $\bar{x}_1, \cdots, \bar{x}_n$ all vanish at g while $\bar{x}_{n+1}$ does not. That completes the proof.

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[2] Although defined quite differently, this embedding seems to be closely related to that constructed in [1].