ON THE HOMOLOGY OF FIBER SPACES

MICHAEL GINSBURG[1]

Let (F, T, X, π) be a fiber space, with fiber F, base X, total space T, and fiber map π. A general problem of great interest is that of computing the homology groups of one of the spaces involved, usually F or T, in terms of the homology groups of the other two spaces and, perhaps, some other invariants of the fiber space. In this paper we show how the Lusternik-Schnirelmann category of X enters into this problem and affects the relations which may exist between the homology groups of F, T, and X.

Our main results are stated as theorems and corollaries in §§3 and 4 of this paper, and are summarized here. Let ΩX denote the space of loops on X. If $\text{cat}(X) \leqq k$, we obtain a spectral sequence, $\dot{E}^r$, which relates $H(F)$ and $H(\Omega X)$ with $H(T)$ and for which the differentials, d^r, and groups, $\dot{E}^r_p$, vanish if r, $p \geqq k$. If $\text{cat}(X) \leqq 2$, we obtain an infinite exact sequence relating $H(X)$, $H(F)$, and $H(T)$ which generalizes the Wang sequence. This allows us to compute the additive structure of $H(\Omega X)$ and to partially determine the Pontryagin ring $H_*(\Omega X)$. We also consider the Leray-Serre spectral sequence of the fiber space and essentially compute all the differentials if $\text{cat}(X) \leqq 2$.

Our method is to replace the chain group of T by a twisted tensor product, $\overline{B}A \tilde{\otimes} C(F)$, where $C(Y)$ denotes the group of chains of Y, $A = C(\Omega X)$, and $\overline{B}A$ is the "bar construction" on A. We then apply certain results of [5] which relate $\text{cat}(X)$ and $\overline{B}A$. The necessary definitions and preliminary material are covered in §§1 and 2, while the proofs of the main theorems are in §5.

Some related results are contained in [6]. In that paper we also obtain some results involving the "category of a map," similar to those obtained here by using the "category of a space."

1. **Fiber spaces.** Let X be a space with base point x_0. Let PX denote the space of Moore paths on X (see [3]). Thus if R^+ denotes the non-negative real numbers and $I_r = [0, r]$ for $r \in R^+$, then

$$PX = \{\alpha_r \mid \alpha_r : I_r \to X, r \in R^+\}.$$

A product, $\alpha_r \cdot \beta_s$, is defined in PX if $\alpha_r(r) = \beta_s(0)$. If we let $EX = \{\alpha_r \in PX \mid \alpha_r(r) = x_0\}$ and $p: EX \to X$ be given by $p(\alpha_r) = \alpha_r(0)$,

Received by the editors February 12, 1963.

[1] This research was partially supported by the Air Force Office of Scientific Research contract number AF 49(638)-568.

then p is a map. $\Omega X = p^{-1}(x_0)$, called the *loop-space* of X, is an associative H-space.

For convenience of notation we will identify the constant paths of PX with the points of X. Thus, the path $\alpha_0: I_0 \to x$ will be denoted simply by x. In this way, x_0 becomes the unit for the multiplication in PX.

Let $\pi: T \to X$ be a map and let

$$U_\pi = \{(\alpha_r, t) \in PX \times T \mid \alpha_r(r) = \pi(t)\}.$$

A *lifting function* for π is a map $\lambda: U_\pi \to T$ such that

1. $\pi\lambda(\alpha_r, t) = \alpha_r(0)$;
2. $\lambda(x, t) = t$ for all $x \in X$.

The lifting function λ is called *weakly transitive* if

$$\lambda(\alpha_r \cdot \beta_s, t) = \lambda(\alpha_r, \lambda(\beta_s, t))$$

whenever $\alpha_r(r) = \beta_s(0) = x_0$ and $(\beta_s, t) \in U_\pi$. We will call (F, T, X, π, λ) a (*weakly transitive*) *fiber space* if λ is a (weakly transitive) lifting function for the map $\pi: T \to X$. $F = \pi^{-1}(x_0)$ is called the fiber.

It is easy to show that our definition of fiber space is equivalent to requiring that π have the strong covering homotopy property for all spaces. Furthermore, every fiber space is fiber-homotopically equivalent to a weakly-transitive one, as noted by Brown [2]. An example of a fiber space which actually admits a weakly transitive lifting function is $(\Omega X, EX, X, p)$.

Let A be a DGA algebra over the ring Z of integers, and let $\overline{B}A$ denote the bar construction on A (see [3] or [4]). The notation involved in the definition of $\overline{B}A$ is

$$\begin{aligned} \overline{A} &= A/Z, \\ \overline{A}^0 &= Z, \\ \overline{A}^k &= \overline{A} \otimes \cdots \otimes \overline{A},\ k \text{ times},\ k \geqq 1, \\ \overline{B}A &= \sum_{k=0}^{k=\infty} \overline{A}^k, \text{ the direct sum.} \end{aligned}$$

As usual, we write $[a_1, \cdots, a_k]$ for $a_1 \otimes \cdots \otimes a_k \in \overline{A}^k$ and $[\ ]$ for the unit of $\overline{A}^0$. $\overline{B}A$ is a chain complex with differential $\bar{d}$ and a gradation given by

$$\text{degree } [a_1, \cdots, a_k] = k + \sum_{i=1}^{i=k} \text{degree } a_i$$

for a_i homogeneous elements of $\overline{A}$. Let $\overline{B}_n A$ denote the elements of $\overline{B}A$ of degree n, and let $\overline{B}^n A = \sum_{k=0}^{k=n} \overline{A}^k$.

Let $C(Y)$ denote the group of normalized singular chains of the space Y. We will adopt the convention that if Y is arcwise connected, $C(Y)$ will mean chains whose vertices are all at the base point of Y. Now let (F, T, X, π, λ) be a weakly-transitive fiber space. $A = C(\Omega X)$ is a DGA algebra [3], while by restriction, λ induces a map $\Omega X \times F \to F$ which in the usual way induces a map $\bar{\lambda}: A \otimes C(F) \to C(F)$. $\bar{\lambda}$ thus defines an A-module structure on $C(F)$, and we will write $\bar{\lambda}(a \otimes b) = a \cdot b$. Let $\overline{B}A \tilde{\otimes} C(F)$ denote the usual tensor product of graded groups, with the twisted differential $\tilde{d}$ defined by

$$\tilde{d}([a_1, \cdots, a_k] \otimes c) = \bar{d}[a_1, \cdots, a_k] \otimes c + (-1)^p [a_1, \cdots, a_k] \otimes \partial c + (-1)^{p + \text{degree } a_k} [a_1, \cdots, a_{k-1}] \otimes a_k \cdot c,$$

for a_i homogeneous elements of $\overline{A}$, $c \in C(F)$, and degree $[a_1, \cdots, a_k] = p$. It is shown in [2] that $\overline{B}A \tilde{\otimes} C(F)$ is a chain complex; the subcomplex $Z \tilde{\otimes} C(F)$ is isomorphic, as a chain complex, to $C(F)$ and will be denoted simply $C(F)$.

Let $\tilde{D}_n = \sum_{k=0}^{k=n} \overline{B}_k A \tilde{\otimes} C(F)$. Then $\tilde{D}_0 = C(F)$, $\tilde{D}_n \subseteq \tilde{D}_{n+1}$, and the subcomplexes $\tilde{D}_i$ form a filtration of $\overline{B}A \tilde{\otimes} C(F)$. This gives rise, in the usual way, to a spectral sequence $\tilde{E}^r$.

THEOREM 1.1. *If (F, T, X, π, λ) is a weakly transitive fiber space and $\pi_1(X) = 0$, then*

(i) *the exact homology sequence of the pair (T, F) is isomorphic to that of the pair $(\overline{B}A \tilde{\otimes} C(F), C(F))$;*

(ii) *for $r \geqq 2$, the spectral sequence $\tilde{E}^r$ is isomorphic to the Leray-Serre spectral sequence (see [8]) of the fiber map π.*

This proposition is merely a résumé of statements in [2].

2. **Category.** Let X^k denote the k-fold cartesian product of X, and let

$$T^k(X) = \{(x_1, \cdots, x_k) \in X^k \mid x_i = x_0 \text{ for some } 1 \leqq i \leqq k\}.$$

Choose x_0^k as the base point of both X^k and $T^k(X)$. Then by definition $\operatorname{cat}(X) \leqq k$ if and only if the diagonal map of X into X^k can be deformed, preserving the base points, into $T^k(X)$. This is equivalent to the classical definition if X is separable, metric, an ANR, and x_0 is a nondegenerate base point in the sense of Puppe [7]. (Classically, $\operatorname{cat}(X) \leqq k$ if X can be covered by k open (or closed) sets each of which is contractible to a point in X.)

Again let $A = C(\Omega X)$, and let $j: \overline{B}^n A \to \overline{B}A$ be the inclusion map.

THEOREM 2.1. *If $\pi_1(X) = 0$ and $\operatorname{cat}(X) \leqq k$, there exists a chain map*

θ: $\bar{B}A \to \bar{B}^{k-1}A$ such that $j_*\theta_*$: $H(\bar{B}A) \to H(\bar{B}A)$ is the identity isomorphism.

PROOF. The author has constructed spaces W and W_{k-1}, $W_{k-1} \subseteq W$, and maps ϕ, ψ, p, and f such that the diagram

$$\begin{tikzcd} \bar{B}A \arrow[r, "\phi"] & C(W) \arrow[r, "p"] & C(X) \arrow[dl, "f"] \\ \bar{B}^{k-1}A \arrow[u, "j"] \arrow[r, "\psi"'] & C(W_{k-1}) \arrow[u, "i"] & \end{tikzcd}$$

is commutative, where i is induced by inclusion. Furthermore ϕ, ψ, and p are chain equivalences (see [**5**, Theorem 3.3 and the proof of Theorem 2.2]). Let ψ^{-1} be any chain inverse to ψ, and define $\theta = \psi^{-1}fp\phi$. θ is then a chain map. Since $(\psi^{-1})_*$ is really the isomorphism inverse to ψ_*, and $i_*f_* = p_*^{-1}$, we have

$$j_*\theta_* = j_*\psi_*^{-1}f_*p_*\phi_* = \phi_*^{-1}i_*f_*p_*\phi_* = \phi_*^{-1}p_*^{-1}p_*\phi_* = \mathrm{id}.$$

It follows from Theorem 2.1 that if $\mathrm{cat}(X) \leqq 2$ and (F, T, X, π, λ) is a weakly transitive fiber space, we can define a map λ_*: $H_p(X \times F) \to H_{p-1}(F)$ as follows. Let $A = C(\Omega X)$ be written as the direct sum $\bar{A} + Z$. Then the map S: $\bar{B}^1A \to A$, defined by $S[a] = a$ and $S[\ ] = 0$, lowers the degree by 1 unit and anti-commutes with the boundary operators (see [**3**]). It is simple to check that $S\theta \otimes 1$: $\bar{B}A \otimes C(F) \to A \otimes C(F)$ has the same property. Then the map $\bar{\lambda}(S\theta \otimes 1)$: $\bar{B}A \otimes C(F) \to C(F)$ induces the desired map λ_* on homology groups. In the next section we will study this map λ_*.

3. **The case** $\mathrm{cat}(X) \leqq 2$. In this section it will always be assumed that (F, T, X, π, λ) is a weakly transitive fiber space with $\pi_1(X) = 0$ and $\mathrm{cat}(X) \leqq 2$. All homology groups considered are supposed to have coefficients in a fixed principal ideal domain, which will not be explicitly displayed in the notation. Let i_*: $H_n(X \times F) \to H_n(X \times F, x_0 \times F)$ be the epimorphism induced by inclusion.

THEOREM 3.1. *There exists an infinite exact sequence*

$$\cdots \to H_n(F) \xrightarrow{f} H_n(T) \to H_n(X \times F, x_0 \times F) \xrightarrow{g} H_{n-1}(F) \to \cdots,$$

where f is induced by inclusion and $g = -\lambda_* i_*^{-1}$.

Let $H(X) \otimes H(F)$ be graded in the usual way, and identify $H(F)$ with $H_0(X) \otimes H(F)$. $H(X) \otimes H(F)$ is a natural subgroup of $H(X \times F)$;

let $\lambda'_* = \lambda_* | H(X) \otimes H(F)$. Clearly $\lambda'_*\lambda'_* = 0$ and λ'_* lowers degree by one unit, so we may consider the homology groups of $H(X) \otimes H(F)$ with this differential. The following corollary follows immediately from Theorem 3.1.

COROLLARY 1. *Suppose $H(X)$ and $H(F)$ are both free. Then $H(T) \approx H(H(X) \otimes H(F))$.*

If λ is the natural lifting function given by path multiplication for the map $p: EX \to X$, then λ_* restricts to the map $\lambda'_*: H_p(X) \otimes H_q(\Omega X) \to H_{p+q-1}(\Omega X)$. According to Theorem 2.1, $H_n(X)$ can be identified with a direct summand of $H_n(\overline{B}^1 A) \approx H_{n-1}(\Omega X)$ for $n \geqq 2$. Thus λ'_* can be considered as defining a multiplication between certain elements of $H_*(\Omega X)$.

COROLLARY 2.
(i) *λ'_* is Pontryagin multiplication;*
(ii) *for $n \geqq 1$, the additive structure of $H_*(\Omega X)$ is computable from the induction formula*

$$H_n(\Omega X) = \sum_{r+s=n+1;\, s<n} H_r(X) \otimes H_s(\Omega X) + \sum_{r+s=n;\, s<n-1} H_r(X) \otimes H_s(\Omega X).$$

PROOF. Part (i) follows trivially from the definitions; part (ii) follows from Theorem 3.1, since EX is contractible, on computing $H_n(\Omega X) \approx H_{n+1}(X \times F, x_0 \times F)$ by the Künneth formula and using the relations

$$H_0(X, x_0) \approx H_1(X, x_0) \approx 0, \qquad H_i(X) \approx H_i(X, x_0) \quad \text{for } i \geqq 1.$$

The formula above was originally shown by G. W. Whitehead to hold if X is a suspension space [**3**].

Let $E^r(\pi)$ denote the Leray-Serre spectral sequence of the fiber map $\pi: T \to X$. If the differentials d^r are all trivial for $r<k$, $E^2_{p,q}(\pi)$ is canonically isomorphic to $E^k_{p,q}(\pi)$. Denote by $\sigma: H_p(X) \otimes H_q(F) \to E^k_{p,q}(\pi)$ the canonical monomorphism of $H_p(X) \otimes H_q(F)$ into $E^2_{p,q}(\pi)$ followed by this isomorphism. Also let $\tau: H_n(F) \to E^k_{0,n}(\pi)$ be the canonical epimorphism.

THEOREM 3.2.
(i) *$d^r: E^r_{p,q}(\pi) \to E^r_{p-r,q+r-1}(\pi)$ is zero if $p \neq r$;*
(ii) *there is a commutative diagram for $p \geqq 2$,*

$$\begin{array}{ccc} H_p(X) \otimes H_q(F) & \xrightarrow{-\lambda'_*} & H_{p+q-1}(F) \\ \downarrow \sigma & & \downarrow \tau \\ E^p_{p,q}(\pi) & \xrightarrow{d^p} & E^p_{0,p+q-1}(\pi). \end{array}$$

REMARK 1. The exact sequence of Theorem 3.1 exists even if (F, T, X, π, λ) is not a weakly transitive fiber space, for Brown [2] has shown that every fiber space is fiber homotopically equivalent to a weakly transitive one. Applying Theorem 3.1 to this latter fiber space provides the sequence. Similarly, part (i) of Theorem 3.2 is valid for any fiber space.

REMARK 2. If $H(X)$ is free, Corollary 2 says that the Pontryagin algebra $H_*(\Omega X)$ is the tensor algebra of $H(X)$ after shifting down by one unit the degrees of elements of $H(X)$. This also follows from a theorem of Bott-Samelson [1], since it is shown in [5] that if $\operatorname{cat}(X) \leqq 2$, the homology suspension is an epimorphism and thus all elements of $H(X)$ are transgressive.

4. **The general case.** If $A \supseteq B$, let $[A, B]^p$ denote the p-fold cartesian power of the pair.

THEOREM 4.1. *Let (F, T, X, π, λ) be a weakly transitive fiber space with $\pi_1(X)=0$ and $\operatorname{cat}(X) \leqq k$. Then there exists a spectral sequence $\dot{E}^r$ such that*

(i) *$\dot{E}^\infty$ is the graded group associated with $H(T)$ under a suitable filtration;*

(ii) *$\dot{E}^1_{p,q} = H_q([\Omega X, b]^p \times F)$, where b is the base point of ΩX;*

(iii) *$d^r = 0$ for $r \geqq k$;*

(iv) *$\dot{E}^\infty_{p,q} = 0$ for $p \geqq k$.*

Let $\overline{B}A \tilde{\otimes} C(F)$ (see §1) be filtered by the $\dot{D}_n = \sum_{i=0}^{i=n} \overline{B}^i A \tilde{\otimes} C(F)$. The $\dot{D}_n$ are subcomplexes of the chain complex $\overline{B}A \tilde{\otimes} C(F)$, and give rise to the spectral sequence $\dot{E}^r$ in the usual way. The proof of Theorem 4.1 is now almost word for word a repetition of the proofs of Theorems 2.1, 2.2, and 3.1 of [5]. As the proofs of these theorems are long and are given in detail in [5], we will omit a repetition of these arguments.

5. **Proof of the theorems.** Let (F, T, X, π, λ) be a weakly transitive fiber space such that $\pi_1(X)=0$. Let $A = C(\Omega X)$, and let $\overline{B}A \otimes C(F)$ denote the usual tensor product of chain complexes. $C(F)$ will be identified with the subcomplex $\overline{B}_0 A \otimes C(F)$. We will consider the following four chain complexes, filtrations, and spectral sequences:

1. $\overline{B}A \tilde{\otimes} C(F)$ with the filtration $\tilde{D}_n$ and spectral sequence $\tilde{E}^r$ defined in §1;

2. $\overline{B}A \otimes C(F)$ with the corresponding filtration

$$D_n = \sum_{k=0}^{k=n} \overline{B}_k A \otimes C(F)$$

and spectral sequence E^r;

3. $\overline{B}A\tilde{\otimes}C(F)/C(F)$ with the filtration $\tilde{R}_n=\tilde{D}_n/\tilde{D}_0$ and spectral sequence $\tilde{E}^r(R)$;

4. $\overline{B}A\otimes C(F)/C(F)$ with the filtration $R_n=D_n/D_0$ and spectral sequence $E^r(R)$.

Note that E^r and $E^r(R)$ are trivial spectral sequences.

The canonical projections $\tilde{\eta}$: $\overline{B}A\tilde{\otimes}C(F)\to\overline{B}A\tilde{\otimes}C(F)/C(F)$ and η: $\overline{B}A\otimes C(F)\to\overline{B}A\otimes C(F)/C(F)$ are filtration preserving chain maps and induce maps $\tilde{\eta}_r$ and η_r of the corresponding spectral sequences.

LEMMA 5.1. *For* $1\leqq r\leqq p$, η_r: $E^r_{p,q}\approx E^r_{p,q}(R)$ *and* $\tilde{\eta}_r$: $\tilde{E}^r_{p,q}\approx\tilde{E}^r_{p,q}(R)$.

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} D_p/D_{p-r} & \xrightarrow{i} & D_{p+r-1}/D_{p-1} \\ \downarrow \eta' & & \downarrow \eta'' \\ R_p/R_{p-r} & \xrightarrow{j} & R_{p+r-1}/R_{p-1} \end{array}$$

where the vertical maps are induced by η and i and j are inclusions. For $p-r\geqq 0$ and $p-1\geqq 0$, that is for $1\leqq r\leqq p$, the vertical maps are isomorphisms of chain complexes. In the induced homology diagram

$$\begin{array}{ccc} H_{p+q}(D_p/D_{p-r}) & \xrightarrow{i_*} & H_{p+q}(D_{p+r-1}/D_{p-1}) \\ \downarrow \eta'_* & & \downarrow \eta''_* \\ H_{p+q}(R_p/R_{p-r}) & \xrightarrow{j_*} & H_{p+q}(R_{p+r-1}/R_{p-1}) \end{array}$$

since η'_* and η''_* are isomorphisms, η''_* maps the image of i_* isomorphically onto the image of j_*. But image $i_*=E^r_{p,q}$, image $j_*=E^r_{p,q}(R)$, and $\eta''_*|E^r_{p,q}=\eta_r$. The result for $\tilde{\eta}$ follows by putting tildes over everything in sight.

Now consider the chain map $j\theta$: $\overline{B}A\to\overline{B}A$ of Theorem 2.1. Let i: $C(F)\to C(F)$ be the identity map. Then the map of graded groups

$$j\theta\otimes i\colon \overline{B}A\otimes C(F)\to\overline{B}A\tilde{\otimes}C(F)$$

induces a map of graded groups

$$\psi\colon \overline{B}A\otimes C(F)/C(F)\to\overline{B}A\tilde{\otimes}C(F)/C(F).$$

LEMMA 5.2. *ψ is a chain equivalence if* $\operatorname{cat}(X)\leqq 2$.

PROOF. If $[a_1,\cdots,a_n]\in\overline{B}_pA$, we can write $j\theta[a_1,\cdots,a_n]=\sum_j[b_j]$ for some $b_j\in A$. Using this and the explicit form of the boundary operators $\tilde{d}$ in $\overline{B}A\tilde{\otimes}C(F)$ and d in $\overline{B}A\otimes C(F)$, an elementary calculation shows that

$$[\tilde{d}(j\theta\otimes i)-(j\theta\otimes i)d][a_1,\cdots,a_n]\otimes c=-\textstyle\sum_j[\ ]\otimes b_j\cdot c\in C(F).$$

Hence $\bar{d}\psi=\psi d$ and ψ is a chain map.

Since ψ is clearly filtration preserving, it induces maps ψ_r: $E^r(R) \to \tilde{E}^r(R)$. Furthermore, since $\pi_1(X)=0$, it is shown in [2] that there are natural isomorphisms

$$\lambda\colon \bar{B}_pA \otimes H_q(F) \approx E^1_{p,q} \quad \text{and} \quad \tilde{\lambda}\colon \bar{B}_pA \otimes H_q(F) \approx \tilde{E}^1_{p,q},$$

where in each case, $\bar{B}A \otimes H(F)$ has the boundary operator $\bar{d}\otimes 1$. A routine calculation shows that the diagram

$$\begin{tikzcd}
\bar{B}_pA \otimes H_q(F) \arrow[r, "j\theta \otimes i_*"] \arrow[d, "\lambda"'] & \bar{B}_pA \otimes H_q(F) \arrow[d, "\tilde{\lambda}"] \\
E^1_{p,q} \arrow[d, "\eta_1"'] & \tilde{E}^1_{p,q} \arrow[d, "\tilde{\eta}_1"] \\
E^1_{p,q}(R) \arrow[r, "\psi_1"] & \tilde{E}^1_{p,q}(R)
\end{tikzcd}$$

is commutative. By Lemma 5.1, η_1 and $\tilde{\eta}_1$ are group isomorphisms for $p\geqq 1$, and so are isomorphisms of complexes for $p\geqq 2$. Since $j_*\theta_*$ is the identity isomorphism by Theorem 2.1, it follows that ψ_2: $E^2_{p,q}(R) \approx \tilde{E}^2_{p,q}(R)$ for all $p\geqq 2$. Since $E^2_{p,q}(R)\approx\tilde{E}^2_{p,q}(R)\approx 0$ for $p\leqq 1$, ψ_2 is an isomorphism. By a standard argument, so is ψ_*. As all the chain complexes considered are free, ψ is a chain equivalence.

Proof of Theorem 3.1. According to Theorem 1.1 we may replace the exact homology sequence of (T, F) by that of $(\bar{B}A \tilde{\otimes} C(F), C(F))$. By Lemma 5.2, the relative group in this sequence is $H(\bar{B}A \otimes C(F), C(F))$, which, by the Künneth theorem, is $H(X\times F, x_0\times F)$. Hence the sequence of Theorem 3.1 exists and f is induced by inclusion.

Let $\bar{\mu}\in H_n(X\times F)$ be represented by the cycle $\mu=\sum_i b_i\otimes c_i$ of $\bar{B}A \otimes C(F)$. Then $\theta(b_i)=\sum_j [b_{ij}]+r_i[\ ]$ for some $b_{ij}\in\bar{A}$ and $r_i\in Z$. By definition, $\lambda_*(\bar{\mu})$ is represented by

$$\tilde{\lambda}(S\theta \otimes 1)(\mu) = \tilde{\lambda}(S \otimes 1)\left[\sum_{ij} [b_{ij}] \otimes c_i + \sum_i r_i[\ ] \otimes c_i\right] = \sum_{ij} b_{ij}\cdot c_i,$$

while $gi_*(\bar{\mu})$ is represented by

$$\bar{d}\psi i(\mu) = \bar{d}(j\theta \otimes 1)(\mu) = (j\theta \otimes 1)d(\mu) - \sum_{ij} [\ ] \otimes b_{ij}\cdot c_i.$$

Since $d(\mu)=0$, $gi_*(\bar{\mu})=-\lambda_*(\bar{\mu})$, and the theorem is proven.

Proof of Theorem 3.2. By Theorem 1.1, we may replace $E^r(\pi)$ by $\tilde{E}^r$. In proving Lemma 5.2 we have shown that ψ_r: $E^r(R)\to\tilde{E}^r(R)$ is an isomorphism for $r\geqq 2$; thus $\tilde{E}^r(R)$ is a trivial spectral sequence. By Lemma 5.1, $\tilde{\eta}_r$: $\tilde{E}^r_{p,q}\to\tilde{E}^r_{p,q}(R)$ is an isomorphism for $p\geqq r$, which implies $\bar{d}^r$: $\tilde{E}^r_{p,q}\to\tilde{E}^r_{p-r,q+r-1}$ is trivial for $p>r$. $\bar{d}^r=0$ for $p<r$ trivially, so $\bar{d}^r=0$ for $r\neq p$.

Letting a generator $\mu \in H_p(X) \otimes H_q(F)$ be represented by $b \otimes c$, where b and c are cycles, we have, as in the proof of Theorem 3.1, $\tau\lambda_*(\mu)$ represented by $\sum_j b_j \cdot c$. On the other hand, $j\theta(b) = \sum_j [b_j]$ is homologous to b since $j_*\theta_*$ is the identity isomorphism, according to Theorem 2.1. Hence μ is also represented by $\sum_j [b_j] \otimes c$. As $\bar{d} \sum_j [b_j] \otimes c = -\sum_j [\ ] \otimes b_j \cdot c \in \tilde{D}_0$, $\sigma(\mu)$ is also represented by $\sum_j [b_j] \otimes c$, while $\bar{d}^p \sigma(\mu)$ has $-\sum_j [\ ] \otimes b_j \cdot c$ as representative. Thus $\bar{d}^p \sigma = -\tau\lambda_*$.

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University of Chicago