EXTREMAL LENGTH OF WEAK HOMOLOGY CLASSES ON RIEMANN SURFACES[1]

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1. The square integrable harmonic differentials on a Riemann surface W form a Hilbert space Γ_h. Let Γ_x be a closed subspace of Γ_h. Let c be a 1-chain on W. There exists a unique element $\psi(c) \in \Gamma_x$ with the property $\int_c \omega = (\omega, \psi(c))$ for all $\omega \in \Gamma_x$. We refer to $\psi(c)$ as the Γ_x-reproducing differential for c. Accola [1] has shown that if c is a cycle, then the extremal length of the homology class of c is equal to the square of the norm of the Γ_h-reproducing differential for c (cf. also [3]). Two specific problems raised by Accola's result are the following. For the important subspaces Γ_x, does the norm of the Γ_x-reproducing differential for a cycle have an extremal length interpretation? Secondly, we may ask for a family of curves associated with a 1-chain c, not necessarily a cycle, whose extremal length gives the norm of the Γ_h-reproducer for c. (By Abel's theorem, the vanishing of the norm of this reproducer implies that ∂c is a principal divisor.)

In the present paper we give an answer to the first question for the subspace Γ_{hse}. Theorem 1 states that an associated geometric configuration is the weak homology class of c.[2]

2. Let Γ_x be a closed subspace of Γ_h such that $\Gamma_x = \bar{\Gamma}_x$. We say that two cycles c_1 and c_2 are Γ_x-homologous, denoted by $c_1 \approx c_2 \pmod{\Gamma_x}$, if $\int_{c_1 - c_2} \omega = 0$ for all $\omega \in \Gamma_x$. Denote the Γ_x-homology class of a cycle c by c^x.

An invariant expression $\rho(z)|dz|$ with ρ a nonnegative and lower semicontinuous function is called a linear density. The ρ-area is

$$A(\rho) = \iint_W \rho^2 dxdy.$$

The ρ-length of a family $\mathfrak{F}$ of arcs is

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[2] Another special case of the first question has been settled by A. Marden. He has shown that a geometric configuration for the subspace Γ_{h0} (notation as in [2]) is the set of relative, i.e., possibly infinite cycles which are weakly homologous to c [*An extremal length problem and the bilinear relation on open Riemann surfaces*, doctoral dissertation, Harvard University, May, 1962].

$$L(\mathfrak{F},\ \rho)=\inf\left\{\int_c \rho|dz| : c\in\mathfrak{F}\right\}.$$

The extremal length of $\mathfrak{F}$ is

$$\lambda(\mathfrak{F}) = \sup_\rho L(\mathfrak{F},\ \rho)^2/A(\rho).$$

LEMMA 1. *Let c be a cycle on W. Let $\lambda(c^x)$ be the extremal length of all cycles Γ_x-homologous to c. Let ψ be the Γ_x-reproducing differential for c. Then $\lambda(c^x)\geqq\|\psi\|^2$.*

PROOF. Let $\rho_0|dz|$ be the linear density $|\psi+i\psi^*|$. Then $A(\rho_0) =\|\psi\|^2$. For $\delta\in c^x$ we have $\int_\delta\rho_0|dz|\geqq|\int_\delta\psi|=|\int_c\psi|=\|\psi\|^2$, and the desired inequality follows. We have used the fact that $\Gamma_x=\bar{\Gamma}_x$ which implies that ψ is real.

3. We shall prove a converse of Lemma 1 for $\Gamma_x=\Gamma_{hse}$. First note that $c_1\approx c_2 \pmod{\Gamma_{hse}}$ if and only if c_1-c_2 is weakly homologous to zero. In fact, $c_1\approx c_2 \pmod{\Gamma_{hse}}$ holds exactly when c_1-c_2 is a dividing cycle (see Theorem V.20D of [2]), which in turn is equivalent to being weakly homologous to zero (see Theorem I.32C, ibid.).

Let Ω be the interior of a compact bordered Riemann surface $\bar{\Omega}$. Let c be a cycle in Ω and ψ_Ω the Γ_{hse}-reproducer for c. Let L_1 be the normal operator for the canonical partition of $\partial\Omega$ (cf. [2]). Corollary 6 of [4] shows that $\psi_\Omega=(2\pi)^{-1}dp^*$ where p is a harmonic function on $W-c$ and satisfies $p=L_1p$ in a boundary neighborhood of Ω. Thus p is constant on each contour β_i of $\bar{\Omega}$ and $\int_{\beta_i}dp^*=0$. If δ is a cycle on Ω then $(2\pi)^{-1}\int_\delta dp$ is an integer equal to the intersection number $\delta\times c$. Furthermore, the Γ_{hse}-reproducer for an open surface W is the limit of ψ_Ω for exhausting canonical subregions $\Omega\rightarrow W$.

In the course of the following proofs we find occasion to use arguments similar to those expressed or implied in Accola [1]. For convenience to the reader we repeat his reasoning in such situations.

LEMMA 2. *Let c be a cycle on a compact bordered Riemann surface and ψ the Γ_{hse}-reproducing differential for c. Then $\lambda(c^{hse})=\|\psi\|^2$.*

PROOF. Denote the bordered surface by $\bar{\Omega}$ and its interior by Ω. Let the contours be $\beta_1, \cdots, \beta_n$. Let $\mathfrak{V}$ be the equivalence class of β_1 in the sense of Accola [1]. That is, $\mathfrak{V}$ is the set of points in $\bar{\Omega}$ which can be joined to a point of β_1 by an arc δ for which $\int_\delta\psi^*$ is an integer. $\mathfrak{V}$ is a closed set which is locally a level curve of a harmonic function. Let $\Omega-\mathfrak{V}=R_1\cup\cdots\cup R_m$ be a decomposition into components. Each R_ν is a finite surface. The points of $\mathfrak{V}$ serve as a piecewise analytic border for R_ν if we allow multiplicities for the prime ends. R_ν together with its boundary shall be denoted by R_ν^*. We may omit the details of this construction but remark that there is an analytic mapping $R_\nu^* \to \bar{\Omega}$ which restricts to the identity on R_ν. By means of this mapping we refer to points of ∂R_ν^* as belonging to $\mathcal{V}$ or $\partial\bar{\Omega}$.

ψ^* is exact on R_ν, say $\psi^* = dp_\nu$, and we know that p_ν extends harmonically to R_ν^*. Since each point of ∂R_ν^* belongs to $\mathcal{V}$ or $\partial\bar{\Omega}$ we see that p_ν is constant on each component of ∂R_ν^*. We adjust p_ν so that the smallest such constant on $\mathcal{V}$ is zero. Now let ρ_ν be the collection of those boundary components of R_ν^* on which $p_\nu = 0$, σ_ν those on which $p_\nu = 1$, and let τ_ν contain the remaining ones. We orient them so that $\partial R_\nu^* = \rho_\nu + \sigma_\nu + \tau_\nu$. The points of ρ_ν and σ_ν belong to $\mathcal{V}$, those of τ_ν belong to $\partial\bar{\Omega} - \mathcal{V}$. Let us show that if τ_ν contains a point t of some β_k then it must contain all of β_k. For $t_1 \in \beta_1$, $\int_{t_1}^t \psi^*$ is not an integer and since $\psi^* = 0$ along β_k, it follows that β_k has a connected neighborhood disjoint from $\mathcal{V}$. This neighborhood must be in R_ν, hence $\beta_k \subset \tau_\nu$.

By means of the mapping $R_\nu^* \to \bar{\Omega}$, we consider σ_ν as a 1-chain on $\bar{\Omega}$ and claim that $c \approx \sum_\nu \sigma_\nu$ (mod Γ_{hse}). Let $\omega \in \Gamma_{hse}$ and assume that ω extends harmonically to $\bar{\Omega}$. Then $\int_c \omega = (\omega, \psi) = \sum_\nu (\omega, dp_\nu^*)$. By partial integration we have $(\omega, dp_\nu^*) = \iint_{R_\nu} dp_\nu \wedge \omega = \int_{\sigma_\nu} \omega + \int_{\tau_\nu} p_\nu \omega$. We have seen that τ_ν is a union of contours $\beta_{\nu_1}, \cdots, \beta_{\nu_k}$ on each of which p_ν is a constant. Since ω is semiexact we obtain $\int_c \omega = \int_{\sum_\nu \sigma_\nu} \omega$. It follows that $c - \sum_\nu \sigma_\nu$ is a dividing cycle.

The function p_ν in R_ν has boundary values 0 on ρ_ν, 1 on σ_ν and constants k_{ν_μ} on β_{ν_μ}, those contours of $\bar{\Omega}$ which make up τ_ν. These constants must satisfy $0 < k_{\nu_\mu} < 1$ in order for the flux condition $\int_{\beta_{\nu_\mu}} dp_\nu^* = 0$ to hold. Consequently, for $s \in (0, 1)$ the level curves $\sigma_\nu(s) = p_\nu^{-1}(s)$ are compact and weakly homologous to σ_ν, except for the finite number of values $s = k_{\nu_\mu}$. Let $\sigma(s) = \sum_\nu \sigma_\nu(s)$. Let ρ be a linear density on Ω. Then for almost all $s \in (0, 1)$

$$L^2(\rho, c^{hse}) \leqq \left(\int_{\sigma(s)} \rho\psi\right)^2 \leqq \int_{\sigma(s)} \rho^2\psi \int_{\sigma(s)} \psi = \|\psi\|^2 \int_{\sigma(s)} \rho^2\psi.$$

Integrating over $s \in (0, 1)$ we obtain

$$L^2(\rho, c^{hse}) \leqq \|\psi\|^2 A(\rho).$$

This, together with the opposite inequality of Lemma 1, completes the proof.

4. Theorem. *Let W be an open Riemann surface. Let ψ be the Γ_{hse}-reproducing differential for a cycle c on W. Then $\|\psi\|^2$ gives the extremal length of all cycles weakly homologous to c.*

PROOF. Let Ω be a canonical subregion of W and ψ_Ω the $\Gamma_{hse}(\Omega)$-reproducing differential for c. Thanks to the above lemmas we have

$$\lambda(c^{hse}) \geqq \|\psi\|^2 = \lim_{\Omega\to W} \|\psi_\Omega\|_\Omega^2 = \lim_{\Omega\to W} \lambda_\Omega(c^{hse}) \geqq \lambda(c^{hse}).$$

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