EXTREME HAMILTONIAN CIRCUITS. RESOLUTION OF THE CONVEX-ODD CASE

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Let n points in the Euclidean plane fall on the boundary B of their convex hull. It is well known that a shortest polygon passing through these points coincides with B. But, it is not known how to explicitly indicate a longest polygon having these n points as vertices. In this paper we do this for the case where n is odd.[1]

THEOREM. *Let*

$$P_1, P_3, P_5, \cdots, P_{2p-1}, P_2, P_4, \cdots, P_{2p-2} \tag{1}$$

be points in the plane which fall on the boundary B of their convex hull in the stated (linear or cyclic) order (accordingly as the points (1) *are or are not collinear). Then $[P_1P_2 \cdots P_{2p-1}]$*[2] *is a longest polygon with* (1) *as vertices; if no three points of* (1) *are collinear, it is the only one.*

PROOF. *Case* I. *Suppose no three points of* (1) *are collinear.* An edge of a polygon intersected by all noncontiguous edges must have the vertices of the polygon alternatively on either side, and is thus an edge of $[P_1P_2 \cdots P_{2p-1}]$. This implies that $[P_1P_2 \cdots P_{2p-1}]$ is the only polygon with each closed edge intersecting every other closed edge.

The symbol $[V_1 \cdots V_{i-1}(V_i \cdots V_j)V_{j+1} \cdots V_n]$ will denote the polygon $[V_1 \cdots V_{i-1}V_jV_{j-1} \cdots V_iV_{j+1} \cdots V_n]$; the operation $[\cdots(\cdots)\cdots]$ will be referred to as an *arcinversion* (cf. [1, p. 180]). Let $h=[R_1 \cdots R_{2p-1}]$ denote any polygon having (1) as vertices and which is distinct from $[P_1 \cdots P_{2p-1}]$. We show that there is an arcinversion which yields a longer polygon. Let i denote the smallest integer such that the closed edge R_iR_{i+1} of h does not intersect at least one of the closed edges $R_1R_2, R_2R_3, \cdots, R_{i-1}R_i$ of h; i of course satisfies $(2<i<2p-1)$. Then, the vertices R_{i-1} and R_i define the following partition of B: $B_1 \cup B_2 \cup \{R_{i-1}, R_i\}$, where B_1 is the component of $B-\{R_{i-1}, R_i\}$ which contains R_1.

Case A. *R_{i-2} and R_{i+1} in the same component of $B-\{R_{i-1}, R_i\}$.*

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[1] *Added in proof.* The remaining case, where n is even, has recently been resolved by the authors (Abstract 611-60, Notices Amer. Math. Soc. 11 (1964), 335).

[2] The symbols for polygons are to be considered cyclic and symmetric.

(i) Suppose i is odd. Then R_{i+1} is in B_1 and the closed edge R_iR_{i+1} does not intersect the closed edge R_1R_2. For, if $R_iR_{i+1}\cap R_1R_2\neq\emptyset$ ($\emptyset$ denotes the empty set), then the closed edge R_iR_{i+1} would intersect each of the closed edges R_1R_2, R_2R_3, $\cdots$, $R_{i-1}R_i$. The arcinversion $[R_1(R_2\cdots R_i)R_{i+1}\cdots R_{2p-1}]$ yields a polygon which is longer than h.

(ii) Suppose i is even. Then R_{i+1} is in B_2 and the closed edge R_iR_{i+1} does not intersect the closed edge R_2R_3. Thus, the arcinversion $[R_1R_2(R_3\cdots R_i)R_{i+1}\cdots R_{2p-1}]$ yields a polygon which is longer than h.

Case B. *R_{i-2} and R_{i+1} in different components of $B-\{R_{i-1}, R_i\}$.* Let C_1 denote the component B_1 or B_2 which contains at most $p-2$ vertices, and C_2 the component which has at least $p-1$ vertices. Let the vertices of h be renumbered consecutively as follows: $h=[S_1S_2\cdots S_{2p-1}]$ with $R_{i-1}R_i=S_1S_2$ or S_2S_1 so that S_3 is in C_1.

Let k denote the number of vertices in C_1. We first show that there is at least one edge of h which has both vertices in C_2. There are at most $2k$ edges incident to the vertices in $C_1\cup\{S_1, S_2\}$ which terminate at vertices in C_2. There are $(2p-1)-(k+2)$ vertices in C_2. Thus, there are at least the following number of edges of h which have both vertices in C_2

$$N=\frac{2((2p-1)-(k+2))-2k}{2}=2p-2k-3.$$

Since $k\leqq p-2$, we have $N\geqq 2p-2(p-2)-3=1$.

Let S_iS_{i+1} denote an edge of h which has both vertices in C_2. Then, either $S_1S_i\cap S_2S_{i+1}\neq\emptyset$ or $S_2S_i\cap S_3S_{i+1}\neq\emptyset$. In the former case the arcinversion $[S_1(S_2\cdots S_i)S_{i+1}\cdots S_{2p-1}]$ yields a polygon which is longer than h, and in the latter case the arcinversion

$$[S_1S_2(S_3\cdots S_i)S_{i+1}\cdots S_{2p+1}]$$

yields a polygon which is longer than h.

REMARK. We note that points (1) satisfying Case I have the property that the longest (also, shortest) polygon can be obtained from any other polygon by a sequence of arcinversions each of which strictly increases (decreases) the length of the polygon to which it is applied (cf. [1, Remark III, p. 181]).

Case II. *Suppose B has support lines passing through at least three points of* (1). If the points of (1) are not all collinear, let P be a point in the interior of the convex hull of (1) and $B(t)$ $(0\leqq t<1)$ a family of strongly convex curves circumscribing B and converging to B as t approaches 1. If the points of (1) are all collinear, let P be a point in one of the open half-planes defined by the line on which the points (1) lie and $B(t)$ $(0 \leqq t < 1)$ a family of strongly convex arcs having P_1 and P_{2p-2} as endpoints, converging to B as t approaches 1, and lying in the closed half-plane which does not contain P. Let $P_i(t)$ be the intersection of $B(t)$ with the ray emanating from P and passing through P_i $(1 \leqq i \leqq 2p-1)$. Then, for each t $(0 \leqq t < 1)$, Case I implies $[P_1(t) \cdots P_{2p-1}(t)]$ is longer than any other polygon $[P_{i_1}(t) \cdots P_{i_{2p-1}}(t)]$. Thus, $[P_1 \cdots P_{2p-1}]$ is a polygon of maximum length.

REMARK. We note that in Case II $[P_1 \cdots P_{2p-1}]$ is not necessarily the only longest polygon. For example, in a set (1) for which $P_1, P_3, P_5, \cdots, P_{2p-1}, P_2, P_4$ are collinear the polygons $[P_1 \cdots P_{2p-1}]$ and $[P_1(P_2P_3P_4)P_5 \cdots P_{2p-1}]$ have the same length.

REFERENCE

1. F. Supnick, *Extreme Hamiltonian lines*, Ann. of Math. (2) **66** (1957), 179–201.

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