# A RELATIVE COHOMOLOGY FOR ASSOCIATIVE ALGEBRAS

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**Introduction.** In this paper we propose a relative cohomology for associative algebras over a *commutative ring*. A relative cohomology for associative algebras over a *field* has been given by Hochschild [2]. In [3] the author has given a cohomology for associative algebras over a commutative ring which is a generalisation of Hochschild's cohomology for associative algebras over a field, a description of which can be found in [1]. In order to be able to show that the relative cohomology proposed here is in the same way a true generalisation of Hochschild's relative cohomology, we require a theorem analogous to [3, Theorem 1]. We are unable to prove such a theorem.

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1. **The tensor product and Hom of graded modules over graded rings.** Let $R = \sum_{p\geqq 0} R^p$ be a graded ring, $A = \sum_{p\geqq 0} A^p$ be a graded right $R$-module and $B = \sum_{p\geqq 0} B^p$ be a graded left $R$-module. Let $F$ be a graded free abelian group generated by the pairs $(a, b)$, where $a \in A^p$, $b \in B^q$ $(p \geqq 0,\ q \geqq 0)$, the pair $(a, b)$ being a homogeneous element of degree $p+q$. Let $G$ be the graded subgroup generated by the homogeneous elements of the form $(a+a',\ b)-(a,\ b)-(a',\ b)$, $(a, b+b')-(a, b)-(a, b')$, and $(a\lambda, b)-(-1)^r(a, \lambda b)$, where $a, a' \in A^p$; $b, b' \in B^q$; $\lambda \in R^r$ $(p \geqq 0, q \geqq 0, r \geqq 0)$. The factor group $F/G$ is a graded abelian group which we call the tensor product of the graded right $R$-module $A$ and the graded left $R$-module $B$ over the graded ring $R$ and denote it by $A \otimes_R B$. We denote the image of the element $(a, b)$ of $F$ in $A \otimes_R B$ by $a \otimes b$. Then $(a+a') \otimes b = a \otimes b + a' \otimes b$, $a \otimes (b+b') = a \otimes b + a \otimes b'$, and $a\lambda \otimes b = (-1)^r a \otimes \lambda b$.

Suppose now that both $A$ and $B$ are graded left $R$-modules. We say that $f: A \to B$ is a graded-$R$-linear map if $f(a+a') = f(a) + f(a')$, and $f(\lambda a) = (-1)^r \lambda f(a)$, where $a, a' \in A$ and $\lambda$ is a homogeneous element of $R$ of degree $r$. The set of all such graded-$R$-linear maps is an abelian group which we denote by $\mathbf{Hom}_R(A, B)$. When $A$ and $B$ are both right $R$-modules, we define $\mathbf{Hom}_R(A, B)$ in an analogous fashion.

The tensor product and **Hom** defined here are different from the usual tensor product and Hom of graded modules over a graded ring inasmuch as the grading of the base ring $R$ is also taken into account. We shall have occasion to use the usual Hom also in the sequel.

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2. **The complex** $\mathcal{B}(U, V)$. By a set-couple [resp. module-couple, algebra-couple] $(P, Q)$ we mean that $P$ is a set [resp. module, algebra] and $Q$ is a subset [resp. submodule, subalgebra] of $P$. By a map [resp. homomorphism] $f: (P, Q) \to (P', Q')$ of set-couples [resp. module-couples, algebra-couples] we understand a map [resp. homomorphism] $f: P \to P'$ such that $f(Q) \subset Q'$.

Let $K$ be a commutative ring with identity $1 \neq 0$. By an algebra we shall mean an associative algebra over $K$ and except when mention is made to the contrary an associative algebra will be understood to possess an identity. A homomorphism of algebras will be understood to map the identity (if there exists one) into the identity.

Let $(\Lambda, \Gamma)$ be an algebra-couple. We construct (cf. [**3**, p. 172]) a differential graded algebra-couple $(U, V)$, where $U = \sum_{n \geqq 0} U_n$, $V = \sum_{n \geqq 0} V_n$, together with a homomorphism of differential graded algebra-couples $\epsilon: (U, V) \to (\Lambda, \Gamma)$, the differential and grading in $(\Lambda, \Gamma)$ being trivial, having the following properties:

(i) For every $n \geqq 0$, $U_n = K(X_n)$ [resp. $V_n = K(Y_n)$], the $K$-free module having the set $X_n$ [resp. $Y_n$] as base.

(ii) An associative multiplication is defined in $U$ [resp. $V$] by $K$-linear extension of the maps of set-couples

$$(X_i \times X_j, Y_i \times Y_j) \to (X_{i+j}, Y_{i+j})$$

for which $(x_i \cdot x_j) \cdot x_k = x_i \cdot (x_j \cdot x_k)$; $x_i \in X_i$, $x_j \in X_j$, $x_k \in X_k$, where $x_i \cdot x_j$ denotes the image of $(x_i, x_j)$ in $X_{i+j}$. In particular, $(U_0, V_0)$ is an associative algebra-couple.

(iii) The restriction of the homomorphism $\epsilon: (U_0, V_0) \to (\Lambda, \Gamma)$ to $(X_0, Y_0)$ is a bijective map which preserves multiplication.

(iv) For each $n \geqq 1$, there is a homomorphism of $K$-module-couples $d_n: (U_n, V_n) \to (U_{n-1}, V_{n-1})$ such that for $n = 1$, the restriction of $d_1$ to $(X_1, Y_1)$ is a bijective map of $(X_1, Y_1)$ onto the kernel-couple $(N_0, L_0)$ of $\epsilon$ and that for $n \geqq 2$, the restriction of $d_n$ to $(X_n, Y_n)$ is a bijective map of $(X_n, Y_n)$ onto the kernel-couple $(N_{n-1}, L_{n-1})$ of $d_{n-1}$. We define $d_0 = 0$.

(v) We have $d_{i+j}(x_i \cdot x_j) = (d_i x_i) \cdot x_j + (-1)^i x_i \cdot (d_j x_j)$; $x_i \in X_i$, $x_j \in X_j$ $(i \geqq 0, j \geqq 0)$, which by $K$-linearity gives an analogous relation when $x_i \in U_i$, $x_j \in U_j$ $(i \geqq 0, j \geqq 0)$.

Let

$$S(U, V) = \sum_{n \geqq -1} S_n(U, V),$$

where

$$S_n(U, V) = U \otimes_V \cdots \otimes_V U, \qquad (n + 2) \text{ factors, } n \geqq -1.$$

If $x=u_0\otimes \cdots \otimes u_{n+1}\in S_n(U, V)$, $n\geqq -1$, the grading in $S(U, V)$ is given by

$$\deg x = n + \sum_{i=0}^{n+1} \deg u_i.$$

We define a map $\tau: S_{n-1}(U, V)\rightarrow S_n(U, V)$, $(n\geqq 0)$ by the relation $\tau(x)=1\otimes x$, where $x\in S_{n-1}(U, V)$. Then $\tau$ is a graded-$V$-left-linear and $U$-right-linear map. For

$$\begin{aligned}\tau(vx) &= 1 \otimes vx = (-1)^{\deg v}\, v \otimes x = (-1)^{\deg v}\, v(1 \otimes x)\\ &= (-1)^{\deg v}\, v\tau(x),\\ \tau(xu) &= 1 \otimes xu = (1 \otimes x)u = \tau(x)u,\end{aligned}$$

where $x\in S_{n-1}(U, V)$, $v$ is a homogeneous element of $V$ and $u\in U$.

We now define two differentials $\partial_r: S_n(U, V)\rightarrow S_n(U, V)$ and $\partial_s: S_n(U, V)\rightarrow S_{n-1}(U, V)$ with the help of $d$ and $\tau$ exactly as in [**3**, p. 166] and set $\partial=\partial_r+\partial_s$. We thus obtain a $K$-linear homogeneous map $\partial: S(U, V)\rightarrow S(U, V)$ of degree $-1$ such that $\partial^2=0$. The explicit formulas for $\partial_r$ and $\partial_s$ are

$$\begin{aligned}&\partial_r(u_0 \otimes \cdots \otimes u_{n+1})\\ &\qquad = du_0 \otimes u_1 \otimes \cdots \otimes u_{n+1}\\ &\qquad\quad + \sum_{i=1}^{n} (-1)^{i+\deg u_0+\cdots+\deg u_{i-1}}\, u_0 \otimes \cdots \otimes du_i \otimes \cdots u_{n+1}\\ &\qquad\quad + (-1)^{n+\deg u_0+\cdots+\deg u_n} u_0\otimes \cdots \otimes u_n \otimes du_{n+1};\\ &\partial_s(u_0 \otimes \cdots \otimes u_{n+1})\\ &\qquad = \sum_{i=0}^{n} (-1)^{i+\deg u_0+\cdots+\deg u_i}\, u_0 \otimes \cdots \otimes u_iu_{i+1} \otimes \cdots \otimes u_{n+1}\end{aligned}$$

for $n\geqq 0$.

Let

$$\mathcal{B}(U, V) = \sum_{n\geqq 0} S_n(U, V) = S(U, V)/S_{-1}(U, V).$$

Since $S_{-1}(U, V)$ is stable for $\partial$, $\partial$ induces a differential (again denoted by $\partial$) in $\mathcal{B}(U, V)$.

3. **The relative cohomology of an algebra.** Let $M$ be a $\Lambda$-bimodule. The homomorphism $\epsilon$: $(U, V)\rightarrow(\Lambda, \Gamma)$ induces over $M$ the structure of a $U$-bimodule such that

$$um = 0 = mu, \qquad m \in M,\ u \in U,\ \deg u > 0,$$

and

$$(du)m = 0 = m(du), \qquad m \in M, \; u \in U \text{ and } \deg u = 1.$$

Let $U^*$ [resp. $V^*$] be the opposite algebra of $U$ [resp. $V$] and let $U^e = U \otimes_K U^*$ [resp. $V^e = V \otimes_K V^*$] denote the enveloping algebra of $U$ [resp. $V$]. Then $M$ can be given the structure of a left $U^e$-module. We denote by $\mathrm{Hom}_{U^e}(\mathfrak{B}(U, V), M)$ the direct sum

$$\sum_{n \geq 0} \mathrm{Hom}_{U^e}(S_n(U, V), M).$$

It can be shown as in [3] that $\mathrm{Hom}_{U^e}(\mathfrak{B}(U, V), M)$ is a complex and the differential $\delta$ is given by $\delta f = f\partial$, where $f \in \mathrm{Hom}_{U^e}(\mathfrak{B}(U, V), M)$.

If we write $U^n = U \otimes_V \cdots \otimes_V U$ ($n$ factors) with $U^0 = V$, there is a natural isomorphism

$$\alpha : \mathbf{Hom}_{V^e}(U^n, M) \to \mathrm{Hom}_{U^e}(S_n(U, V), M)$$

given by

$$(\alpha f)(u_0 \otimes \cdots \otimes u_{n+1}) = u_0 f(u_1 \otimes \cdots \otimes u_n) u_{n+1},$$

where $f \in \mathbf{Hom}_{V^e}(U^n, M)$ and $u_0, \cdots, u_{n+1} \in U$. We now have an isomorphism

$$\mathrm{Hom}_{U^e}(\mathfrak{B}(U, V), M) \approx \sum_{n \geq 0} \mathbf{Hom}_{V^e}(U^n, M).$$

If $f \in \mathbf{Hom}_{V^e}(U^n, M)$, then $\delta f = g + h$, where $g \in \mathbf{Hom}_{V^e}(U^n, M)$ and $h \in \mathbf{Hom}_{V^e}(U^{n+1}, M)$ such that

$$\begin{aligned} g(u_1 \otimes \cdots \otimes u_n) &= \sum_{i=1}^{n} (-1)^{i + \deg u_1 + \cdots + \deg u_{i-1}} f(u_1 \otimes \cdots \otimes du_i \otimes \cdots \otimes u_n), \\ h(u_1 \otimes \cdots \otimes u_{n+1}) &= u_1 f(u_2 \otimes \cdots \otimes u_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^{i + \deg u_1 + \cdots + \deg u_i} f(u_1 \otimes \cdots \otimes u_i u_{i+1} \otimes \cdots \otimes u_{n+1}) \\ &\quad + (-1)^{n+1+\deg u_1 + \cdots + \deg u_{n+1}} f(u_1 \otimes \cdots \otimes u_n) u_{n+1}. \end{aligned}$$

DEFINITION. The $K$-module $H^*(\mathrm{Hom}_{U^e}(\mathfrak{B}(U, V), M))$ is called the relative cohomology module of the $K$-algebra-couple $(\Lambda, \Gamma)$ with coefficients in the $\Lambda$-bimodule $M$ and is denoted by $H^*(\Lambda, \Gamma, M)$.

4. **Interpretations of $H^0(\Lambda, \Gamma, M)$ and $H^1(\Lambda, \Gamma, M)$.** The relative $n$-cochains with coefficients in $M$ for $n = 0, 1, 2, 3$ are as follows (cf. [3, p. 182]):

(i) A relative 0-cochain is an element of $\mathbf{Hom}_{V^e}(V, M)$ and may be identified with an element $m \in M$ for which $\gamma m = m\gamma$ for all $\gamma \in \Gamma$.

(ii) A relative 1-cochain is an element of $\mathbf{Hom}_{V^e}(U_0, M)$ and is determined by a map $\chi : \Lambda \to M$ such that

$$\chi(\gamma\lambda) = \gamma\chi(\lambda), \qquad \chi(\lambda\gamma) = \chi(\lambda)\gamma, \qquad \lambda \in \Lambda,\ \gamma \in \Gamma.$$

(iii) A relative 2-cochain is an element of degree 2 of $\mathbf{Hom}_{V^e}(U \otimes_V U, M) + \mathbf{Hom}_{V^e}(U, M)$ and is determined by two maps

$$\chi_1\colon \Lambda \times \Lambda \to M,$$
$$\chi_2\colon N_0 \to M,$$

such that $\chi_1(\gamma\lambda_1, \lambda_2) = \gamma\chi_1(\lambda_1, \lambda_2)$, $\chi_1(\lambda_1\gamma, \lambda_2) = \chi_1(\lambda_1, \gamma\lambda_2)$, $\chi_1(\lambda_1, \lambda_2\gamma) = \chi_1(\lambda_1, \lambda_2)\gamma$, $\chi_2(\gamma n_0) = \gamma\chi_2(n_0)$, $\chi_2(n_0\gamma) = \chi_2(n_0)\gamma$, $\chi_2(l_0) = 0$; $\lambda_1, \lambda_2 \in \Lambda$, $\gamma \in \Gamma$, $n_0 \in N_0$, $l_0 \in L_0$.

(iv) A relative 3-cochain is determined by four maps

$$\pi_1\colon \Lambda \times \Lambda \times \Lambda \to M,$$
$$\pi_2\colon \Lambda \times N_0 \to M,$$
$$\pi_3\colon N_0 \times \Lambda \to M,$$
$$\pi_4\colon N_1 \to M,$$

such that $\pi_1(\gamma\lambda_1, \lambda_2, \lambda_3) = \gamma\pi_1(\lambda_1, \lambda_2, \lambda_3)$, $\pi_1(\lambda_1\gamma, \lambda_2, \lambda_3) = \pi_1(\lambda_1, \gamma\lambda_2, \lambda_3)$, $\pi_1(\lambda_1, \lambda_2\gamma, \lambda_3) = \pi_1(\lambda_1, \lambda_2, \gamma\lambda_3)$, $\pi_1(\lambda_1, \lambda_2, \lambda_3\gamma) = \pi_1(\lambda_1, \lambda_2, \lambda_3)\gamma$, $\pi_2(\gamma\lambda, n_0) = \gamma\pi_2(\lambda, n_0)$, $\pi_2(\lambda\gamma, n_0) = \pi_2(\lambda, \gamma n_0)$, $\pi_2(\lambda, n_0\gamma) = \pi_2(\lambda, n_0)\gamma$, $\pi_2(\lambda, l_0) = 0$, $\pi_3(\gamma n_0, \lambda) = \gamma\pi_3(n_0, \lambda)$, $\pi_3(n_0\gamma, \lambda) = \pi_3(n_0, \gamma\lambda)$, $\pi_3(n_0, \gamma\lambda) = \pi_3(n_0, \lambda)\gamma$, $\pi_3(l_0, \lambda) = 0$, $\pi_4(\gamma n_1) = \gamma\pi_4(n_1)$, $\pi_4(n_1\gamma) = \pi_4(n_1)\gamma$, $\pi_4(l_1) = 0$; $\lambda, \lambda_1, \lambda_2, \lambda_3 \in \Lambda$, $\gamma \in \Gamma$, $n_0 \in N_0$, $n_1 \in N_1$, and $l_0, l_1 \in L_1$.

Similarly, a relative 4-cochain is determined by eight maps satisfying a number of relations.

The calculations show that $H^0(\Lambda, \Gamma, M)$ is isomorphic to the submodule of the $K$-module $M$ consisting of elements $m \in M$ for which $\lambda m = m\lambda$ for every $\lambda \in \Lambda$. Hence $H^0(\Lambda, \Gamma, M)$ coincides with $H^0(\Lambda, M)$ defined in [3].

We shall call a $K$-homomorphism $f\colon \Lambda \to M$ a crossed $\Gamma$-homomorphism of $\Lambda$ into $M$ if (i) $f(\lambda_1\lambda_2) = \lambda_1 f(\lambda_2) + f(\lambda_1)\lambda_2$ for $\lambda_1, \lambda_2 \in \Lambda$; (ii) $f(\gamma\lambda) = \gamma f(\lambda)$, $f(\lambda\gamma) = f(\lambda)\gamma$ for $\gamma \in \Gamma$, $\lambda \in \Lambda$. The condition (ii) is equivalent to $f(\gamma) = 0$ for $\gamma \in \Gamma$. We shall call a crossed $\Gamma$-homomorphism $f\colon \Lambda \to M$ a principal $\Gamma$-homomorphism if there exists an element $m \in M$ such that $f(\lambda) = \lambda m - m\lambda$, $\lambda \in \Lambda$. Such an element $m$ must necessarily satisfy the relation $\gamma m = m\gamma$ for every $\gamma \in \Gamma$. The calculations then show that $H^1(\Lambda, \Gamma, M)$ is isomorphic to the quotient of the $K$-module of crossed $\Gamma$-homomorphisms of $\Lambda$ into $M$ by the submodule of principal $\Gamma$-homomorphisms of $\Lambda$ into $M$. There is a natural homomorphism

$$H^1(\Lambda, \Gamma, M) \to H^1(\Lambda, M),$$

where $H^1(\Lambda, M)$ is the 1-cohomology module defined in [3]. This homomorphism is actually a monomorphism, since a principal $K$-

homomorphism which is also a crossed $\Gamma$-homomorphism is a principal $\Gamma$-homomorphism.

5. **Interpretation of** $H^2(\Lambda, \Gamma, M)$. Let $\beta$: $E \to \Lambda$ be an onto homomorphism of algebras with kernel $M$ and let $\rho: \Gamma \to E$ be a homomorphism of algebras such that (i) $\beta\rho$ is the identity map over $\Gamma$, (ii) $\rho$ can be extended to a map $\bar{\rho}: \Lambda \to E$ for which $\beta\bar{\rho}$ is the identity map over $\Lambda$ and $\bar{\rho}(\gamma\lambda) = \rho(\gamma)\bar{\rho}(\lambda)$, $\bar{\rho}(\lambda\gamma) = \bar{\rho}(\lambda)\rho(\gamma)$, $\lambda \in \Lambda$, $\gamma \in \Gamma$. We say that the triple $(E, \beta, \rho)$ is a relative extension of the algebra-couple $(\Lambda, \Gamma)$ with kernel $M$. We note that $M$ is a subalgebra of $E$ not necessarily having an identity. Two relative extensions $(E', \beta, \rho)$ and $(E', \beta', \rho')$ of the algebra-couple $(\Lambda, \Gamma)$ with the same kernel $M$ are said to be equivalent if there exists an isomorphism $\phi: E \to E'$ such that $\beta = \beta'\phi$ and $\phi\rho = \rho'$.

A relative extension $(E, \beta, \rho)$ is said to be special if the product of any two elements of the kernel $M$ is zero. In this case $\beta$ induces a $\Lambda$-bimodule structure over $M$.

THEOREM 1. *Let* $(\Lambda, \Gamma)$ *be an algebra-couple and let* $M$ *be a* $\Lambda$*-bimodule. Then there exists a natural one-one correspondence between the relative cohomology module* $H^2(\Lambda, \Gamma, M)$ *and the set of equivalence classes of special relative extensions of the algebra-couple* $(\Lambda, \Gamma)$ *with kernel* $M$ *which induce over* $M$ *the given* $\Lambda$*-bimodule structure.*

The proof is similar to that of [**3**, Theorem 4].

6. **Interpretation of** $H^3(\Lambda, \Gamma, M)$. Let $A$ be an algebra not necessarily having an identity. Let $(E, \beta, \rho)$ be a relative extension of the algebra-couple $(\Lambda, \Gamma)$ with kernel $A$. We do not suppose that the product of any two elements of $A$ is zero. Let $M_A$ denote the algebra of bimultiplications of $A$ and let $P_A$ denote the quotient algebra of exterior bimultiplications of $A$ [**3**, p. 197]. Since $A$ is a two-sided ideal in $E$, the map which assigns to every element $e$ of $E$ the inner bimultiplication of $E$ induced by $e$ gives a homomorphism of algebras $\nu: E \to M_A$. Since $A$ is mapped into the subalgebra of inner bimultiplications, $\nu$ induces a homomorphism of algebras $\theta: \Lambda \to P_A$. If we compose $\nu$ with the homomorphism $\rho: \Gamma \to E$, we get a homomorphism of algebras $\sigma: \Gamma \to M_A$. Since $\rho$ can be extended to a map $\bar{\rho}: \Lambda \to E$ for which $\beta\bar{\rho} = \mathrm{id}_\Lambda$, and $\bar{\rho}(\gamma\lambda) = \rho(\gamma)\bar{\rho}(\lambda)$, $\bar{\rho}(\lambda\gamma) = \bar{\rho}(\lambda)\rho(\gamma)$ where $\lambda \in \Lambda$, $\gamma \in \Gamma$, it follows that $\sigma$ can be extended to a map $\bar{\sigma}: \Lambda \to M_A$ for which $\zeta\bar{\sigma} = \theta$, and $\bar{\sigma}(\gamma\lambda) = \sigma(\gamma)\bar{\sigma}(\lambda)$, $\bar{\sigma}(\lambda\gamma) = \bar{\sigma}(\lambda)\sigma(\gamma)$ where $\lambda \in \Lambda$, $\gamma \in \Gamma$, $\zeta$ being the natural homomorphism of $M_A$ onto $P_A$. We see that a relative extension determines two homomorphisms of algebras $\theta$ and $\sigma$ which satisfy the properties just described. Both $\theta$ and $\sigma$ are regular homomorphisms [**3**, p. 199].

Conversely, suppose we are given two regular homomorphisms $\theta: \Lambda \to P_A$ and $\sigma: \Gamma \to M_A$ which are such that it is possible to define a map $\bar{\sigma}: \Lambda \to M_A$ having the following properties: (i) $\zeta\bar{\sigma} = \theta$, (ii) the restriction of the map $\bar{\sigma}: \Lambda \to M_A$ to $\Gamma$ is the given homomorphism $\sigma: \Gamma \to M_A$, and (iii) $\bar{\sigma}(\gamma\lambda) = \sigma(\gamma)\bar{\sigma}(\lambda)$, $\bar{\sigma}(\lambda\gamma) = \bar{\sigma}(\lambda)\sigma(\gamma)$ where $\lambda \in \Lambda$, $\gamma \in \Gamma$. Does there exist a relative extension $(E, \beta, \rho)$ of the algebra-couple $(\Lambda, \Gamma)$ with kernel $A$ which determines the two regular homomorphisms $\theta$ and $\sigma$? We shall associate with the pair $(\theta, \sigma)$ an element of the relative cohomology module $H^3(\Lambda, \Gamma, C_A)$, where $C_A$ is the bicentre [**3**, p. 198] of $A$. We shall call this element the obstruction of $(\theta, \sigma)$ and shall denote it by $\xi_{(\theta,\sigma)}$.

Choose a map $\bar{\sigma}: \Lambda \to M_A$ having the properties given above. Then $\bar{\sigma}(0) = 0$ and $\bar{\sigma}(1) = 1$, the identity bimultiplication. We note that the quotient algebra $A/C_A$ is isomorphic to the algebra of inner bimultiplications of $A$, which is the kernel of the natural homomorphism $\zeta: M_A \to P_A$. We identify them and define two maps

$$\bar{\chi}_1: \Lambda \times \Lambda \to A/C_A,$$

$$\bar{\chi}_2: N_0 \to A/C_A,$$

such that

$$\bar{\chi}_1(\lambda_1, \lambda_2) = \sigma(\lambda_1\lambda_2) - \sigma(\lambda_1)\sigma(\lambda_2),$$

$$\bar{\chi}_2(n_0) = \sum_1 k_i\sigma(\lambda_i),$$

where $\lambda_1$, $\lambda_2$, $\lambda_i \in \Lambda$, $n_0 = \sum_i k_i(\lambda_i)$, $\sum_i k_i\lambda_i = 0$. Now $A$ and $A/C_A$ are $M_A$-bimodules and the natural map $\mu: A \to A/C_A$ which associates with every element of $A$ the inner bimultiplication of $A$ induced by it, is a homomorphism of $M_A$-bimodules. Thanks to the homomorphism $\sigma: \Gamma \to M_A$ we can consider $A$ and $A/C_A$ as $\Gamma$-bimodules and $\mu$ as a homomorphism of $\Gamma$-bimodules. We can now define two maps

$$\chi_1: \Lambda \times \Lambda \to A$$

$$\chi_2: N_0 \to A$$

such that $\mu\chi_1 = \bar{\chi}_1$, $\mu\chi_2 = \bar{\chi}_2$. It is easy to verify that the maps $\chi_1$ and $\chi_2$ satisfy all the six relations which the two maps $\chi_1$ and $\chi_2$, which determine a relative 2-cochain, satisfy although here $A$ is not a $\Lambda$-bimodule. We now define four maps

$$\pi_1: \Lambda \times \Lambda \times \Lambda \to C_A,$$

$$\pi_2: \Lambda \times N_0 \to C_A,$$

$$\pi_3: N_0 \times \Lambda \to C_A,$$

$$\pi_4: N_1 \to C_A,$$

exactly as in [3, p. 200] in terms of $\chi_1$ and $\chi_2$. It is easy to verify that these four maps satisfy all the relations necessary to determine a relative 3-cochain with coefficients in the $\Lambda$-bimodule $C_A$. It can be shown as in [3, Theorem 5] that this relative 3-cochain is a relative 3-cocycle and that it determines an element of $H^3(\Lambda, \Gamma, C_A)$ which is independent of the choice of the map $\bar{\sigma}: \Lambda \to M_A$. This element is called the obstruction of $(\theta, \sigma)$. We are now in a position to answer the question raised at the beginning of this section.

THEOREM 2. *The pair of regular homomorphisms* $(\theta, \sigma)$ *is induced by a relative extension if and only if the obstruction* $\xi_{(\theta,\sigma)} = 0$.

The proof is similar to that of [3, Theorem 6].

Finally, the $K$-module $H^3(\Lambda, \Gamma, M)$ can be interpreted in terms of the pairs of regular homomorphisms $(\theta, \sigma)$.

THEOREM 3. *Let* $(\Lambda, \Gamma)$ *be an algebra-couple and let $M$ be a* $\Lambda$*-bimodule. Let $f$ be a relative* 3*-cocycle of* $(\Lambda, \Gamma)$ *with coefficients in $M$. Then there exist an algebra $A$ having $M$ as its bicentre and two regular homomorphisms* $\theta: \Lambda \to P_A$ *and* $\sigma: \Gamma \to M_A$ *satisfying the conditions* (i), (ii), *and* (iii) *described above such that $\theta$ induces over $M$ the given* $\Lambda$*-bimodule structure and that $f$ is the obstruction of the pair* $(\theta, \sigma)$.

The proof is similar to that of [3, Theorem 8], but one has to start with the algebra

$$L = U_0 + U_0 \otimes_V U_0 + \cdots + U_0 \otimes_V \cdots \otimes_V U_0 + \cdots$$

instead of the algebra $L$ in [3, p. 205], in which the tensor products were taken over $K$.

REMARK. It would be interesting to investigate whether the relative cohomology proposed in this paper reduces to the absolute cohomology given in [3] when we take for $\Gamma$ the image of $K$ in $\Lambda$ under the natural homomorphism $K \to \Lambda$ which maps the identity of $K$ into the identity of $\Lambda$. If we denote the image of $K$ in $\Lambda$ by $K$ itself, it is evident that an absolute cochain of $\Lambda$ [3, p. 182] is not a relative cochain of the algebra-couple $(\Lambda, K)$, since the maps which determine it need not satisfy any relations of the type described in §4. Yet the interpretations of $H^0(\Lambda, \Gamma, M)$ and $H^1(\Lambda, \Gamma, M)$ show that $H^0(\Lambda, K, M) \approx H^0(\Lambda, M)$ and $H^1(\Lambda, K, M) \approx H^1(\Lambda, M)$. Since the relative cohomology has been described in terms of the canonically constructed algebra-resolutions $U$ and $V$ of $\Lambda$ and $\Gamma$ respectively, the first task, as already indicated in the Introduction, is to construct a category of objects $(U, V, \Lambda, \Gamma, \epsilon)$ (cf. [3, Chapter II]) and to prove theorems analogous to [3, Theorems 1, 2]. The second task will be to see whether we can take $V=\Gamma$, when $\Gamma$ is $K$-projective. If this is found possible, we shall have shown that the relative cohomology does reduce to the absolute cohomology when we take $\Gamma=K$.

## References


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# DIRECT PROOF OF THE BASIC THEOREM ON MULTIPARTITE PARTITIONS

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In what follows all small latin letters denote non-negative integers, while $N$ and $N_{ms}$ are $j$-partite numbers, i.e. vectors or row-matrices of $j$ dimensions whose components are non-negative integers. In particular, $N=(n_1, n_2, \cdots, n_j)$. We write $q_k(N)$ for the number of partitions of $N$ into just $k$ parts and $r_k(N)$ for the number of partitions of $N$ into just $k$ different parts.

Let $\pi=\pi(k)$ be a partition of $k$ into $h(1)$ parts 1, $h(2)$ parts 2 and so on, so that $k=\sum_m mh(m)$. We write

$$H(\pi) = \prod_m \left\{h(m)!m^{h(m)}\right\}^{-1}$$

where, as usual, $0!=1$, and $D(\pi, N)$ for the number of solutions of

$$N = \sum_m \sum_{s=1}^{h(m)} mN_{ms}, \tag{1}$$

where the order of the $N_{ms}$ is relevant. Clearly

$$D(\pi, N) = \prod_{i=1}^{j} D(\pi, n_i). \tag{2}$$

Again, for $|X|<1$,

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