The very short proof of 4.1 depends on the nontrivial result of Hewitt and Zuckerman [3, Theorem 8.10] that a proper H-semigroup always has a semicharacter f such that $0<f<1$. The use of this fact is avoided in the following argument, suggested by the referee: let G be a linear semigroup, H its group of quotients, and G' the set of x in H such that $x^n \in G$ for some $n \in N$. In addition to being positively ordered and archimedean, G' is also naturally ordered and hence, by a theorem of Hölder and Huntington (see [1]), G' is isomorphic to a subsemigroup of R^+; so is G.

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REPRESENTATIVE SETS AND DIRECT SUMS

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1. Let (X, Λ, μ) be a complete measure space and denote by N the collection of all locally μ-null subsets of X, i.e., $E \in N$ if and only if $\mu(E \cap F)=0$ for all μ-summable sets F. For E and F in Λ we write $E \equiv F$ if and only if $E \Delta F \in N$ where $E \Delta F=(E-F) \cup (F-E)$. The relation $E \equiv F$ is clearly an equivalence relation.

J. von Neumann showed that if μ is σ-finite there exists a mapping ρ of Λ into Λ with the following properties:

(1) $\rho(E) \equiv E$;

(2) $E \equiv F$ implies $\rho(E)=\rho(F)$;

(3) $\rho(\emptyset)=\emptyset$, $\rho(X)=X$ ($\emptyset$ = the empty set);

(4) $\rho(E \cap F)=\rho(E) \cap \rho(F)$;

(5) $\rho(E \cup F)=\rho(E) \cup \rho(F)$.

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However, von Neumann's published proof in [3] applies only when $X=R^n$ and μ is the Lebesgue measure. An independent proof for a general σ-finite measure space was later given by D. Maharam [2]. By a still different proof, using Banach algebra methods, A. and C. Ionescu Tulcea showed in [1] that the result also holds when X is locally compact and μ is a positive Radon measure.

When this result holds we will say that (X, Λ, μ) has property (M). The purpose of this note is to show that property (M) is equivalent to X being the direct sum of μ-summable subsets $\{Y_\alpha: \alpha\in A\}$ plus possibly a locally μ-null set. §2 contains some preliminary remarks on the contraction of μ and direct sums. §3 contains the main result. For further results about the material in §2 we refer to the paper by A. C. Zaanen [5]. In this connection we also mention I. E. Segal's fundamental paper on localizable spaces [4]. The author wishes to thank the referee for several useful suggestions incorporated in this paper.

2. The contraction of μ is defined on Λ by $\mu_c(E)=\sup \mu(F)$ over all $F\subset E$ with $\mu(F)<\infty$. It can easily be shown that μ_c is a measure on Λ and that the μ_c-measurable subsets of X are just the μ-measurable subsets Λ, i.e., the Carathéodory extension of μ_c yields no new μ_c-measurable sets. A set $E\in\Lambda$ is μ_c-null if and only if E is locally μ-null. Thus, the space (X, Λ, μ_c) is complete if and only if (X, Λ, μ) is complete. This proves the following:

LEMMA 1. *(X, Λ, μ) has property* (M) *if and only if (X, Λ, μ_c) has property* (M).

We will say that X is a direct sum of μ-summable sets $\{Y_\alpha: \alpha\in A\}$ whenever $X-Y=\bigcup_{\alpha\in A} Y_\alpha$, where Y is either a locally μ-null set with $\mu(Y)=\infty$ or $Y=\emptyset$, where the sets $\{Y_\alpha\}$ are mutually disjoint μ-summable sets with positive μ-measure and where every μ-summable set E is the union, except for a μ-null set, of at most a countable number of the sets $\{E\cap Y_\alpha\}$.

We will assume that $\mu(X)\neq 0$ and that X contains some μ-summable set with positive measure. Thus, if X is a direct sum then A is not empty.

LEMMA 2. *X is a direct sum of μ-summable sets if and only if X is a direct sum of μ_c-summable sets.*

PROOF. Suppose that X is a direct sum of μ-summable sets $\{Y_\alpha: \alpha\in A\}$. Then $Y=X-\bigcup_{\alpha\in A} Y_\alpha$ is a μ_c-null set. By adding Y, which may be empty, to some set Y_{α_0} we have that $X=(\bigcup_{\alpha\neq\alpha_0} Y_\alpha)\cup(Y_{\alpha_0}\cup Y)$ is a direct sum of μ_c-summable sets.

Now, assume that X is a direct sum $X=\bigcup_{\alpha\in A} X_\alpha$ of μ_c-summable sets $\{X_\alpha:\alpha\in A\}$. (The set Y does not appear since locally μ_c-null sets are μ_c-null.) By Lemma 10.1 of [5] there exists for each $\alpha\in A$ a μ-summable set $Y_\alpha\subset X_\alpha$ such that $\mu(Y_\alpha)=\mu_c(X_\alpha)$. Thus $X_\alpha-Y_\alpha$ is locally μ-null. If X_α happens to be μ-summable we choose $Y_\alpha=X_\alpha$ otherwise we have $\mu(X_\alpha-Y_\alpha)=\infty$. Theorem 8.6 of [5] implies that $Y=\bigcup_{\alpha\in A}(X_\alpha-Y_\alpha)$ is measurable and μ_c-null. Hence Y is locally μ-null and $\mu(Y)=\infty$ or $Y=\emptyset$.

Let E be any μ-summable subset of X. Then E is μ_c-summable and $\mu(E)=\mu_c(E)$. By hypothesis there exists a countable number of sets $\alpha=\alpha_1, \alpha_2, \cdots$ such that $\mu_c(E-\bigcup_{i=1}^\infty(E\cap X_{\alpha_i}))=0$. Since $\mu(E\cap Y_\alpha)=\mu_c(E\cap X_\alpha)$ we have

$$\mu\left(E-\bigcup_{i=1}^\infty(E\cap Y_{\alpha_i})\right)=\mu_c\left(E-\bigcup_{i=1}^\infty(E\cap X_{\alpha_i})\right)=0.$$

This proves that X is a direct sum of μ-summable sets $\{Y_\alpha:\alpha\in A\}$.

3. We can now state and prove the main result:

THEOREM. (X, Λ, μ) *has property* (M) *if and only if X is a direct sum of μ-summable sets* $\{Y_\alpha:\alpha\in A\}$.

PROOF. From Lemmas 1 and 2 we can see that the statement of the theorem is equivalent to the following statement: (X, Λ, μ_c) has property (M) if and only if X is a direct sum of the μ_c-summable sets $\{X_\alpha:\alpha\in A\}$. This is the statement we will prove.

Suppose that $X=\bigcup_{\alpha\in A}X_\alpha$ is a direct sum of μ_c-summable sets. By the result of D. Maharam [2] there exists for each $\alpha\in A$ a mapping ρ_α of $\Lambda\cap X_\alpha$ into $\Lambda\cap X_\alpha$ satisfying (1)–(5). Using in general the axiom of choice, choose such a ρ_α for each $\alpha\in A$. Define ρ on Λ by $\rho(E)=\bigcup_{\alpha\in A}\rho_\alpha(E\cap X_\alpha)$. It is a consequence of Theorem 8.6 of [4] that $\rho(E)$ is measurable and that $\mu_c[\rho(E)\Delta E]=0$. Thus ρ satisfies (1); ρ clearly satisfies (2)–(5).

To prove the other half of the theorem we assume that (X, Λ, μ_c) has the property (M). Let ρ be one of the mappings of Λ into Λ satisfying (1)–(5). Let E^* denote the class of all sets $F\in\Lambda$ such that $\mu_c(E\Delta F)=0$. Following Zaanen [5, p. 178] we consider all collections $\mathfrak{F}=\{X_\alpha^*\}$ corresponding to sets $X_\alpha\in\Lambda$ for which $0<\mu_c(X_\alpha)<\infty$ and $\mu_c(X_\alpha\cap X_{\alpha'})=0$ for $\alpha\neq\alpha'$. The assumption that X contains some μ-summable set with positive measure implies that the family $\{\mathfrak{F}\}$ is not empty. The family $\{\mathfrak{F}\}$ is partially ordered by inclusion and it is clear that every chain has a least upper bound. Thus, by Zorn's lemma there exists a maximal element. Let $\mathfrak{F}_m=\{X_\alpha^*:\alpha\in A\}$ be a fixed maximal element.

Consider the sets $\rho(X_\alpha)$. For each $\alpha \in A$, $0 < \mu_c[\rho(X_\alpha)] < \infty$ and $\rho(X_\alpha) \cap \rho(X_{\alpha'}) = \rho(X_\alpha \cap X_{\alpha'}) = \emptyset$ for $\alpha \neq \alpha'$ by (3). If E is an arbitrary μ_c-summable set then by Theorem 8.4 of [4], $\mu_c[E \cap \rho(X_\alpha)] > 0$ for an at most countable number of indices $\alpha = \alpha_1, \alpha_2, \cdots$ and $\mu_c[E - \bigcup_{i=1}^\infty (E \cap \rho(X_{\alpha_i}))] = 0$. This shows that

$$E \cap \left[X - \bigcup_{\alpha \in A} \rho(X_\alpha) \right] = E - \bigcup_{\alpha \in A} [E \cap \rho(X_\alpha)]$$

is a subset of a μ_c-null set for each μ_c-summable set E. Since (X, Λ, μ_c) is assumed to be complete, this proves that $X - \bigcup_{\alpha \in A} \rho(X_\alpha)$ is a μ_c-null set and hence measurable. Now write $Z_{\alpha_0} = \rho(X_{\alpha_0}) \cup [X - \bigcup_{\alpha \in A} \rho(X_\alpha)]$ and $Z_\alpha = \rho(X_\alpha)$ for $\alpha \neq \alpha_0$. Then it is clear that $X = \bigcup_{\alpha \in A} Z_\alpha$ is a direct sum of μ_c-summable sets $\{Z_\alpha : \alpha \in A\}$ (α_0 is an arbitrary element of the index set A). This proves the theorem.

This theorem combined with Theorems 9.2 and 10.2 of [5], gives the following results:

COROLLARY. *If* (X, Λ, μ) *has the property* (M) *then the measure* μ_c *is localizable or, equivalently,* μ *has the Radon-Nikodym property.*

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