A THEOREM ON MAXIMUM MODULUS[1]

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Introduction. If D is a domain in the plane of complex numbers, then every analytic function achieves its maximum modulus only at the boundary. We phrase this by asserting that if f is analytic in D, and f is not constant, and if $x \in \overline{D}$ has the property that

$$\limsup_{x} |f| = \sup_{D} |f|,$$

then $x \in \delta(D)$, the boundary of D. If x is the only point for which the above identity holds, then x is *the peak point for f in D*. If x is the peak point for a bounded analytic function in D, then x is *a peak point* of D. We are interested in knowing which boundary points of D can be peak points.

W. Rudin [1] defines a boundary point x of a domain D as a *removable boundary point* if every function bounded and analytic in D can be continued at x. All boundary points which are not removable are *essential.*

We shall show that a point is a peak point of D if and only if it is an essential boundary point.

LEMMA 1. *If D is a simply connected domain and $x \in \delta(D)$ is linearly accessible from the interior, then there is a schlicht mapping f of D into the open unit disc such that* $\lim_{z \to x} f(z) = 1$ *and* $\limsup_{\xi} |f| < 1$ *if* $\xi \in \delta(D)$, $\xi \neq x$.

PROOF. Let $(x, x+\alpha]$ be a line segment contained in D. Then the function

$$f(z) = g^{-1}(2 + 4(\alpha^{-1}(z - x))^{1/2}),$$

where $g(z) = z + 1/z$, $|z| < 1$, is a schlicht function with the properties:

1. $\lim_{z \to x} f(z) = 1$,
2. $|x - y| > \epsilon$, $y \in D \Rightarrow 1 - |f(y)| > \eta = \eta(\epsilon, D) > 0$,

as we shall show. Let $x_i \to x$, $x_i \in D$ and $|y - x| > \epsilon$. If we examine the mapping $z \to f_1(z) = \alpha^{-1}(z - x)$, it is a linear mapping taking $[x, x+\alpha]$ onto $[0, 1]$. $\alpha^{-1}(x_i - x) \to 0$, $|f_1(y) - 0| > \epsilon/|\alpha|$. Since D, and thus $f_1(D)$, is simply connected, we can apply a square root, with $(+1)^{1/2} = +1$, and get $f_2(z) = (f_1(z))^{1/2}$, $(\alpha^{-1}(x_i - x))^{1/2} \to 0$, and

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$$|f_2(y) - 0| = |(f_1(y))^{1/2}| = (|f_1(y) - 0|)^{1/2} > \left(\frac{\epsilon}{|\alpha|}\right)^{1/2} = \epsilon_1.$$

We note that each point of $(0, 1]$ is an interior point of $f_2(D)$, so that each point of $[-1, 0)$ is an exterior point of $f_2(D)$. Thus, $\mathrm{Cl}[f_2(D)]$ meets $[-1, 0]$ only at 0. Set $\eta_1 = \eta_1(\epsilon, D) = d([-1, 0], f_2(D) - N(0, \epsilon_1)) > 0$. Then $d(f_2(y), [-1, 0]) \geqq \eta_1$. Setting $f_3(z) = 2+4(f_2(z))$, we have $f_3(x_i) \to 2$ and $d(f_3(y), [-2, 2]) \geqq 4\eta_1$. We now observe that for our function g, $g(z) \to 2$ iff $z \to 1$ and $|z| \leqq 1-\epsilon \Rightarrow d(g(z), [-2, 2]) > \epsilon^2$.[2] Thus $f(x_i) = g^{-1}(f_3(x_i)) = g^{-1}(2 + 4(\alpha^{-1}(x_i - x))^{1/2}) \to 1$ and $|f(y)| = |g^{-1}(f_3(y))| < 1 - (4\eta_1)^{1/2}$ so that $1 - |f(y)| > (4\eta_1)^{1/2} = \eta(\epsilon, D) > 0$, as promised.

LEMMA 2. *The set of peak points of any domain is closed.*

PROOF. Let a_i be the peak point of f_i in D, $i = 1, 2, \cdots$, and let $a_i \to a$. We assume that $\sup_D |f_i| = 1$. Let N_i be a neighborhood of a_i, $i = 1, 2, \cdots$, with $\mathrm{diam}(N_i) \to 0$ and pairwise disjoint. Since a_i is a peak point of f_i, f_i is bounded away from 1 off N_i, so that for an appropriate integer m_i, we have

$$\sup_{z \in D - N_i} |f_i^{m_i}(z)| < 4^{-i}.$$

Set $g_i(z) = f_i^{m_i}(z)$. We will now generate (inductively) a sequence $\{b_i\}$ such that

1. $|b_i| \leqq 2$,
2. $\sum_{i=1}^{\infty} b_i g_i$ is bounded and analytic in D,
3. $\sup_{N_n} |\sum_{i=1}^{n} b_i g_i| = 2 - 1/n$.

We take $b_1 = 1$. Given $b_1, \cdots, b_{n-1}$, we find b_n as follows:

Set $h_n(\xi) = \sup_{N_n} |b_1 g_1 + \cdots + b_{n-1} g_{n-1} + \xi \cdot g_n|$, $|\xi| \leqq 2$. Then $h_n(0) = \sup_{N_n} |b_1 g_1 + \cdots + b_{n-1} g_{n-1}| < \sum_{i=1}^{n-1} 2 \cdot 4^{-i} < 2/3$. We now choose a point y_n such that $g_n(y_n) > 1 - (1/2n)$, and set

$$c_n = 2\, \frac{b_1 g_1(y_n) + \cdots + b_{n-1} g_{n-1}(y_n)}{g_n(y_n)} \cdot \left| \frac{g_n(y_n)}{b_1 g_1(y_n) + \cdots + b_{n-1} g_{n-1}(y_n)} \right|.$$

Then

$$\begin{aligned} h_n(c_n) &\geqq |b_1 g_1(y_n) + \cdots + b_{n-1} g_{n-1}(y_n) + c_n g_n(y_n)| \\ &\geqq |c_n g_n(y_n)| > 2 - \frac{1}{n}, \end{aligned}$$

[2] This is because the image under g^{-1} of the circle $x^2 + y^2 = c^2$ is the ellipse $(u^2/a^2) + (v^2/b^2) = 1$, where $a = c + 1/c$, $b = c - 1/c$. If $c = 1 - \epsilon$ then the image of $x^2 + y^2 < (1-\epsilon)^2$ is the exterior of the indicated ellipse. The closest approach of this set to the line $[-2, 2]$ is at the vertex, where the distance is $\epsilon^2/(1-\epsilon)$.

since $\sum_{i=1}^{n-1} b_i g_i(y_n)$ and $c_n g_n(y_n)$ have the same argument. Since $h_n(0) < 2/3 < 2-1/n < h_n(c_n)$, there is a point b_n in the disc $|z| \leqq 2$ such that $h_n(b_n) = 2-1/n$. For $z \in N_n$, we then have

$$\left| \sum_{i=1}^{\infty} b_i g_i(z) \right| \leqq \left| \sum_{i=1}^{n} b_i g_i(z) \right| + \sum_{i=n+1}^{\infty} |b_i| \cdot |g_i(z)|$$

$$< 2 - \frac{1}{n} + \sum_{i=n+1}^{\infty} 2 \cdot 4^{-i} = 2 - \frac{1}{n} + \frac{2}{3} \cdot 4^{-n}.$$

Similarly,

$$\sup_{z \in N_n} \left| \sum_{i=1}^{\infty} b_i g_i(z) \right| > 2 - \frac{1}{n} - \frac{2}{3} \cdot 4^{-n}.$$

For $z \in D - \bigcup_{i=1}^{\infty} N_i$, we have

$$\left| \sum_{i=1}^{\infty} b_i g_i(z) \right| < \sum_{i=1}^{\infty} 2 \cdot 4^{-i} = 2/3.$$

Thus $\left| \sum_{i=1}^{\infty} b_i g_i(z) \right| < 2$ for all $z \in D$. If K is a compact subset of D, then K meets only finitely many N_i, so that the series converges uniformly and absolutely on K. Thus $f(z) = \sum_{i=1}^{\infty} b_i g_i(z)$ is analytic. Finally, it is clear that $|f(z)|$ is close to 2 only inside N_i, for large i, that is, only around a.

LEMMA 3. *If D is any domain, the boundary points linearly accessible from the interior are dense in $\delta(D)$.*

PROOF. Let $x \in \delta(D)$, $\epsilon > 0$. Let $y \in D$, $|y-x| < \epsilon$. Let x_1 be the point of $\delta(D) \cap [x, y]$ lying nearest to y. Then $|x - x_1| < \epsilon$ and x_1 is linearly accessible.

THEOREM 1. *If D is simply connected, and its boundary consists of more than one point, then every boundary point is a peak point.*

PROOF. Direct consequence of Lemmas 1, 2, 3.

LEMMA 4. *If $x \in \delta(D)$ and the component of x in D', $K(x) \neq \{x\}$, then x is a peak point of D.*

PROOF. Let $y \in K(x)$, $y \neq x$. Let K_1 be a sub-continuum of $K(x)$, containing y but not containing x. There is a conformal mapping ϕ of K_1' onto the open unit disc, and ϕ is a homeomorphism around x. Then $\phi(K(x)')$ is a simply connected domain, and each linearly accessible point of $\delta(\phi(K(x)'))$ is a peak point *of the kind described in Lemma* 1. If we now restrict our functions to $\phi(D)$, each of these points is still a peak point, though the point may no longer be linearly accessible. Now, by Lemma 2, x is a peak point.

LEMMA 5. *If $K(D)$ denotes the closure of the union of those components of $\delta(D)$ which are not single points, then every point in $K(D)$ is a peak point of D.*

PROOF. Clear from Lemmas 4 and 2.

DEFINITION. A set S is a *Painlevé null set* (called a *p-null set*) if the algebra of bounded analytic functions on S' consists of the constants alone.

DEFINITION. A point $x \in \delta(D)$ is called a *p-essential boundary point* if for each $\epsilon > 0$, $N(x, \epsilon) \cap \delta(D)$ is not a p-null set.

LEMMA 6. *Let $x \in \delta(D)$ and $x \notin K(D)$. Then if x is a p-essential boundary point, x is a peak point of D.*

PROOF. Let N be a neighborhood of x in which $\delta(D)$ is totally disconnected. Since the p-essential boundary points form a perfect set, let $x_i \to x$ be a sequence of p-essential boundary points. Let N_i be a sequence of neighborhoods with $N_i \subset N$, $x_i \in N_i$ no two N_i intersecting, and $\operatorname{diam}(N_i) \to 0$. Let M_i be open with $x_i \in M_i \subset \overline{M}_i \subset N_i$ for each i. Then $K_i = \overline{M}_i \cap \delta(D)$ is not a p-null set and thus we can find f_i such that f_i is analytic on K_i' (including ∞) and f_i is not constant. We can assume that $\sup_{K_i'} |f| = 1$. Then $\sup_{N_i'} |f_i| < 1$, so we can choose a sequence of integers m_i so that

$$\sup_{N_i'} |f_i^{m_i}| < 4^{-i}.$$

Set $g_i(z) = f_i^{m_i}(z)$. Since D' is nowhere dense in N_i, $\sup_D |g_i| = 1$, and $\sup_{D-N_i} |g_i| < 4^{-i}$.

Using the same technique as in Lemma 2, we choose b_n so that

1. $|b_n| \leqq 2$,
2. $\sup_{N_i} \left|\sum_{i=1}^{n} b_i g_i\right| = 2 - 1/n$,
3. $f(z) = \sum_{i=1}^{\infty} b_i g_i(z)$ is bounded and analytic in D.

Furthermore, we deduce that

$$2 - \frac{1}{n} - \frac{2}{3} \cdot 4^{-n} < \sup_{N_n} |f| < 2 - \frac{1}{n} + \frac{2}{3} \cdot 4^{-n},$$

and $\sup_{D - \cup_i N_i} |f| < 2/3$. Thus, f has a peak at x.

THEOREM 2. *If x is a p-essential boundary point, then x is a peak point.*

PROOF. If $x \in K(D)$, this is true by Lemma 5; otherwise by Lemma 6.

THEOREM 3. *x is a peak point iff x is an essential boundary point.*

PROOF. By a theorem of W. Rudin [1], x is an essential boundary point iff x is a p-essential boundary point. From the remarks above, every $x \in K(D)$ is a peak point of a function which has no limit at x. Thus, no point of $K(D)$ is removable. If $x \in \delta(D) - K(D)$ and x is removable, then each f is continuable there. Since $\delta(D)$ is nowhere dense around x, this extension does not change the maximum modulus nor the lim sup at x. By the maximum modulus theorem, $f(x)$ is not the maximum. By continuity, x is not a peak for any f.

It would be interesting to know when x is the peak point of a bounded analytic function f which has a limit at x, or whose modulus has a limit at x. It is clear from previous remarks that every essential boundary point is the peak point of a function which does *not* have these limits. This question is open.

REFERENCE

1. W. Rudin, *Some theorems on bounded analytic functions*, Trans. Amer. Math. Soc. **78** (1955), 333–342.

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