ON RINGS ON RINGS[1]

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This paper presents a solution to a popular problem in the subject of rings of analytic functions. In the late 1940's it was shown that two domains D_1 and D_2 in the complex plane were conformally equivalent (to within a certain equivalence relation) iff the rings $B(D_1)$ and $B(D_2)$ of all bounded analytic functions defined on them were algebraically isomorphic.[3] In the case of annuli, two are conformally equivalent iff the ratio of the radii of one equals the same ratio for the other. It follows that this ratio must be contained somewhere in the algebraic structure of the ring. The problem is to find it. More exactly,

1. **Problem.** Let $\mathfrak{R}$ be a ring which is known to be isomorphic with the ring of bounded analytic functions on an annulus $A=\{z \mid \rho_1<|z|<\rho_2\}$, where ρ_1 and ρ_2 are not known. From the ring $\mathfrak{R}$, deduce the number ρ_2/ρ_1.

Actually, the problem solved here is not the original one. That one dealt with the algebra of *all* analytic functions. As we shall see later in the paper, this original problem has a solution also, and this solution is somewhat simpler. Furthermore, the solution in this simpler case extends to other, more complicated, domains.

To solve Problem 1, we will let ϕ be the isomorphism mapping $B(A)$ onto $\mathfrak{R}$, and will denote elements of $B(A)$ by f, g, h and elements of $\mathfrak{R}$ by a, b, c, d, e (e is the multiplicative identity). Let $\mathbf{1}\in B(A)$ be the function identically equal to 1 on A. Then clearly $e=\phi(\mathbf{1})$, and $ne=\phi(n\mathbf{1})$, so that $\pm(m/n)e=\phi(\pm(m/n)\cdot\mathbf{1})$. $-e$ has two square roots in $\mathfrak{R}$, one being the image of $i\cdot\mathbf{1}$, the other the image of $-i\cdot\mathbf{1}$. It is algebraically impossible to distinguish between these, since $\mathfrak{R}$ has an automorphism which takes one into the other (corresponding to the mapping $f\rightarrow\bar{f}$ in $B(A)$). Thus, we choose one root of $-e$ and make it

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[1] This problem has gone under this title for some years and, some feel, was invented to make the title possible. The author has heard the invention of the problem credited to a few sources, most frequently to Professor Melvin Henriksen of Purdue. Professor Henriksen reports that the problem is due to Professor P. C. Rosenblum of Minnesota but (proudly) takes full credit for the title.

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[3] Actually, if $B(D_1)\cong B(D_2)$, then D_1 and D_2 are either conformally or anti-conformally equivalent, after the removable boundary points (see [1]) have been suppressed. In the case of an annulus, neither of these limitations has an effect.

correspond to $i\cdot 1$; denote it as ie. Then $\phi((r_1+r_2i)1)=r_1e+r_2ie$ for all rational r_1, r_2, with possibly a consistent error in the sign of the imaginary term.[4] Note now that $\alpha\in\overline{R}(f)$ [the closed range of f] iff $f-\alpha 1$ has no inverse in $B(A)$, i.e., iff $\phi(f)-\alpha e$ has no inverse in $\mathfrak{R}$, *i.e.*, iff α is in the spectrum $\sigma(\phi(f))$ of $\phi(f)$. Thus, if $a\in\mathfrak{R}$, we know for each rational r_1, r_2 whether $r_1+ir_2\in\overline{R}(\phi^{-1}(a))$ by knowing whether $(r_1+ir_2)e-a$ has an inverse in $\mathfrak{R}$. Therefore, if $\phi^{-1}(a)$ is not a constant, we know $\overline{R}(\phi^{-1}(a))$. Specifically, if $\rho(a)=\sup\{|\alpha|\mid\alpha\in\sigma(a)\}$, then $\rho(a)$ is also the maximum modulus (hereinafter abbreviated MM) of $\phi^{-1}(a)$. At this point, if we could find the function z, we would be finished, for $\mathrm{MM}(z)\mathrm{MM}(z^{-1})=\rho_2\cdot(1/\rho_1)$. Actually, we cannot hope to do better than finding $\alpha z^{\pm 1}$, $\alpha\neq 0$ which is algebraically indistinguishable from z. In any case, if $\phi^{-1}(a)(z)=\alpha z^{\pm 1}$, $\alpha\neq 0$, then $\rho(a)\cdot\rho(a^{-1})=\rho_2/\rho_1$. We now require an algebraic characterization of $\alpha z^{\pm 1}$.

2. **Lemma.** *Let A be an annulus, and let $f\in B(A)$. Suppose that*

$1°$ *$f^{-1}\in B(A)$, where $f^{-1}(z)\cdot f(z)=1$, all $z\in A$.*

$2°$ *f is schlicht.*

$3°$ *For every $g\in B(A)$, either (*) or (**) holds:*

$$\mathrm{MM}(f\cdot g)=\mathrm{MM}(f)\cdot\mathrm{MM}(g), \tag{*}$$

$$\mathrm{MM}(f^{-1}\cdot g)=\mathrm{MM}(f^{-1})\cdot\mathrm{MM}(g). \tag{**}$$

Then $f(z)=\alpha z^{\pm 1}$ for some $\alpha\neq 0$.

Proof. Let $K=\mathrm{MM}(f)$, $k=\mathrm{MM}(f^{-1})$. If every point in the boundary of $R(f)$ has modulus K or k^{-1}, then since f is schlicht, $R(f)$ is an annulus conformal with A, and the conclusion is a well-known result.

We shall, therefore, assume that $x\epsilon\delta(R(f))$, $k^{-1}<|x|<K$, and obtain a contradiction to 3°, *i.e.*, we shall find a function $g\in B(A)$ such that $\mathrm{MM}(g)=1$, but $\mathrm{MM}(f\cdot g)<K$, $\mathrm{MM}(f^{-1}\cdot g)<k$.

First let g_1 be any function analytic in $R(f)$ with a peak at x.[5] Without loss of generality, we can assume that $\mathrm{MM}(g_1)=1$. Let r, ϵ be so chosen that $(|x|-r)^{-1}<k-\epsilon$, $(|x|+r)<K-\epsilon$. Then $\sup\{|g_1(z)|\mid |z-x|\geqq r,\ z\in A\}=\eta<1$. Set $g(z)=g_1(f(z))$. Then g is analytic in A and $\mathrm{MM}(g)=1$. For $z\in A$ with $|x-f(z)|<r$,

$$|f(z)g(z)|=|f(z)|\cdot|g(z)|\leqq(K-\epsilon)\cdot 1,$$
$$|f^{-1}(z)g(z)|=|f^{-1}(z)|\cdot|g(z)|\leqq(k-\epsilon)\cdot 1.$$

[4] This is the last mention we make of this possible error. If $ie=\phi(-i1)$, then an analysis like the one in the remainder of the paper will show that Theorem 3 still holds.

[5] For a definition of peak of a function, see [1].

For $z \in A$, $|x - f(z)| \geqq r$,

$$|f(z)g(z)| = |f(z)| \cdot |g(z)| \leqq K \cdot \eta,$$
$$|f^{-1}(z)g(z)| = |f^{-1}(z)| \cdot |g(z)| \leqq k \cdot \eta.$$

Thus

$$\mathrm{MM}(fg) \leqq \max(K - \epsilon, K\eta) < K = \mathrm{MM}(f)\mathrm{MM}(g),$$
$$\mathrm{MM}(f^{-1}g) \leqq \max(k - \epsilon, k\eta) < k = \mathrm{MM}(f^{-1})\mathrm{MM}(g).$$

Therefore, 3° is violated, which establishes the contradiction.

We need only show that the properties 1°, 2°, 3° on f follow from purely algebraic conditions on $\phi(f)$, and the problem is solved. 1° and 3° are clearly algebraic. We note that if f is not schlicht, then there is some complex rational $r_1 + r_2 i$ such that f takes the value twice, unless f is a constant function. Thus, if $K > k^{-1}$ and if for each $r_1 + r_2 i$ with $k^{-1} < |r_1 + r_2 i| < K$, $f - (r_1 + r_2 i)1$ has only one zero, then f is schlicht. It follows that

3. **Theorem.** *Let $\mathfrak{R}$ be a ring which is algebraically isomorphic with $B(A)$, the ring of bounded analytic functions on the annulus*

$$A = \{z \mid \rho_1 < |z| < \rho_2\}.$$

Then there is an element $a \in \mathfrak{R}$ satisfying 1°, 2°, 3° *below, and for any such element, $\rho(a) \cdot \rho(a^{-1}) = \rho_2/\rho_1$.*

1° $a^{-1} \in B(A)$.

2° *$(\rho(a^{-1}))^{-1} < \rho(a)$ and for every complex rational $r_1 + r_2 i$ with $(\rho(a^{-1}))^{-1} < |r_1 + r_2 i| < \rho(a)$ we have $(a - (r_1 e + r_2 i e))\mathfrak{R}$ is a maximal ideal in $\mathfrak{R}$.*

3° *For every $b \in \mathfrak{R}$ either* (*) *or* (**):

(*) $$\rho(ab) = \rho(a)\rho(b),$$

(**) $$\rho(a^{-1}b) = \rho(a^{-1})\rho(b).$$

Proof. Clearly $\phi(z)$ meets the three conditions. Further, if $\phi(f)$ meets the conditions, then $f(z) = \alpha z^{\pm 1}$, and the conclusion follows.

4. **The original problem.** Let $\mathfrak{R}$ be a ring which is known to be isomorphic with the ring $A(D)$ of all analytic functions on an unknown domain D. Given $\mathfrak{R}$, find a conformal or anti-conformal image of D.

In this case, the restriction concerning removable boundary points is unnecessary. We note, in fact, that the spectrum of an element in $A(D)$ is the actual range of the corresponding function, rather than its closure, and our methods will only yield the closure. If we actually knew all the irrational constant functions, then we could obtain the actual spectrum. We will use a method of W. Rudin to obtain these constant functions. Denote, then, the closure of the spectrum of an element a by $\bar{\sigma}(a)$. We see that if there is a nonconstant $f_0 \in A(D)$ which is bounded, then $z \in R(z\cdot 1+f_0-w\cdot 1)$, where $w \in R(f_0)$. Also,

$$\bigcap_{n>0} \overline{R}\left(z\cdot 1 + \frac{1}{n} f_0 - \frac{1}{n} w\cdot 1\right)$$

is exactly the point z. Thus, if $a \in \mathfrak{R}$ is an element with no complex rational in its spectrum, then a must be the image of $z\cdot 1$ for some irrational z. The value of z is, in fact the only point in $\bigcap \bar{\sigma}(a+b-\alpha e)$, where the intersection is taken over all $b \in \mathfrak{R}$ with at least two complex rationals in $\sigma(b)$, and all complex rationals $\alpha \in \sigma(b)$. Thus, for any $a \in \mathfrak{R}$, we can get $\sigma(a)$, which is the range of the corresponding function. Therefore, if a is schlicht, $\sigma(a)$ is conformal or anti-conformal with D.

In case the domain D has no nonconstant bounded analytic functions, this method collapses completely, since $\bar{\sigma}(a+b-\alpha e)$ is always the whole plane.

Bibliography

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