ON THE SEQUENCE OF FOURIER COEFFICIENTS

P. L. SHARMA

1. Let $f(x)$ be a function which is integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and is defined outside this by periodicity. Let the Fourier series of $f(x)$ be

$$(1.1) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \tfrac{1}{2}a_0 + \sum_{1}^{\infty} A_n(x),$$

then the conjugates series of (1.1) is

$$(1.2) \qquad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{1}^{\infty} B_n(x).$$

Mohanty and Nanda [2] have proved that if

$$(1.3) \quad f(x+t) - f(x-t) - l = \psi(t) = o\left\{\left(\log \frac{1}{t}\right)^{-1}\right\}, \qquad t \to 0,$$

and a_n and b_n are $O(n^{-r})$, $0<r<1$, then the sequence $\{nB_n(x)\}$ is summable $(C, 1)$ to the value l/π.

DEFINITION. The series $\sum a_n$ with the sequence of partial sums $\{S_n\}$ is said to be summable by a regular row-infinite matrix[1] method of summation or summable A if

$$(1.4) \qquad \sigma = \sum_{n=1}^{\infty} a_{n,m} s_n$$

exists and is finite.

We obtain another method of summation viz., $A \cdot (C, 1)$ by superimposing the method A on the Cesàro mean of order one.

The object of this paper is to prove the following theorem.

THEOREM. *If*

$$(1.5) \qquad \psi(t) = o\left\{\left(\log \frac{1}{t}\right)^{-1}\right\}, \qquad \text{as } t \to 0$$

and for some r *with* $0<r<1$,

$$(1.6) \qquad \lim_{n\to\infty} \sum n^r \left| a_{n,m} - a_{n+1,m} \right| = 0,$$

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[1] Here $A = (a_{n,m})$, where $\sum_{n\to\infty} |a_{n,m}| < \infty$, $m = 1, 2, 3, \cdots$.

then the sequence $\{nB_n(x)\}$ *is summable* $A\cdot(C, 1)$ *to the sum* l/π.

It may be noted here that the regular matrices which satisfy the condition (1.6) with $r=0$ are called strongly regular [1].

2. **Proof of the theorem.** If we denote the $(C, 1)$-transform of the sequence $\{nB_n(x)\}$ by τ_n, we have after Mohanty and Nanda [2],

$$\tau_n - \frac{l}{\pi} = \frac{1}{\pi}\int_0^1 \psi(t)\left[\frac{\sin nt}{nt^2} - \frac{\cos nt}{t}\right]dt + o(1) \tag{2.1}$$

by Riemann-Lebesgue theorem.

On account of the regularity of the A method of summation, we need only to prove that

$$I = \frac{1}{\pi}\sum_{\lim_{n\to\infty}} a_{n,m}\int_0^1 \psi(t)g_n(t)dt = o(1), \tag{2.2}$$

where

$$g_n(t) = \frac{\sin nt}{nt^2} - \frac{\cos nt}{t}\,. \tag{2.3}$$

We require the following inequalities which can be easily obtained by expanding sine and cosine in powers of n and t:

$$g_n(t) = O(n^2t) \tag{2.4}$$

and

$$g_n(t) = O(t^{-1}). \tag{2.5}$$

It is also known [3] that

$$\sum_{\nu=1}^{n} \frac{\sin \nu}{\nu} = O(1). \tag{2.6}$$

Now

$$\begin{aligned} |I| &\leqq \left|\frac{1}{\pi}\sum a_{n,m}\left\{\int_0^{n^{-1}} + \int_{n^{-1}}^{n^{-r}} + \int_{n^{-r}}^{1}\right\} g_n(t)\psi(t)dt\right| \\ &= \left|\frac{1}{\pi}\sum a_{n,m}(P+Q+R)\right|. \end{aligned} \tag{2.7}$$

Using (2.4), we get

$$|P| = O(n^2)\int_0^{n^{-1}} o(1)O(t)dt = o(1).$$

Also, with (1.5) and (2.5), we write

$$|Q| = o\left(\int_{n^{-1}}^{n^{-r}} \frac{dt}{t \log 1/t}\right)$$
$$= o(1).$$

Thus the first two terms in (2.7) can be made as small as we please by choosing n sufficiently large.

With the help of (2.3) and (2.5), we write

$$M_\nu(t) = g_1(t) + \cdots + g_\nu(t) = \frac{1}{t^2}\sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu} - \frac{1}{t}\sum_{\nu=1}^{n} \cos \nu t$$
$$= O\left(\frac{1}{t^2}\right) - \frac{1}{t} D_\nu(t) = O\left(\frac{1}{t^2}\right),$$

where $D_\nu(t)$ is the Dirichlet Kernel for convergence of Fourier series, and it is known that $D_\nu(t) = O(1/t)$. It is easy to see that $\sum |a_{n,m}| < \infty$ and $\sum n^r |a_{n,m} - a_{n+1,m}| < \infty$ imply that $n^r a_{n,m} \to 0$, hence using Abel transformation we write,

$$\left|\frac{1}{\pi}\sum_{1}^{n} a_{n,m} R\right| = \left|\frac{1}{\pi}\sum_{1}^{n} a_{n,m} \int_{n^{-r}}^{1} \psi(t)[M_n(t) - M_{n-1}(t)]dt\right|$$
$$\leqq \left|\frac{1}{\pi}\sum_{1}^{n-1} (a_{n,m} - a_{n+1,m}) \int_{n^{-r}}^{1} \psi(t) M_n(t) dt\right|$$
$$+ \left|\frac{1}{\pi}\sum_{2}^{n} a_{n,m} \int_{n^{-r}}^{(n-1)^{-r}} \psi(t) M_n(t) dt\right| + o(1),$$
$$= L_1 + L_2, \quad \text{say.}$$
$$L_1 = O\left\{\sum_{1}^{n-1} |a_{n,m} - a_{n+1,m}| \int_{n^{-r}}^{1} \frac{1}{t^2} dt\right\}$$
$$= O\left\{\sum_{1}^{n-1} (n^{-r} - 1) |a_{n,m} - a_{n+1,m}|\right\}$$
$$= o(1), \quad \text{by (1.6).}$$
$$L_2 = O\left\{\sum_{2}^{n} |a_{n,m}| \int_{n^{-r}}^{(n-1)^{-r}} \frac{1}{t^2} dt\right\}$$
$$= O\left\{\sum_{2}^{n} |a_{n,m}| [(n-1)^r - n^r]\right\}$$
$$= o(1), \qquad \text{for } 0 < r < 1,$$

since $(a_{n,k})$ is a regular row-infinite matrix and $\{(n-1)^r-n^r\}$ is a null sequence.

This completes the proof of the theorem.

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References

1. G. G. Lorentz, *Direct theorems on methods of summability*, Canad. J. Math. 1 (1949), 305–319.

2. R. Mohanty and M. Nanda, *On the behaviour of Fourier coefficients*, Proc. Amer. Math. Soc. 5 (1954), 79–84.

3. E. C. Titchmarsh, *Theory of functions*, p. 440.

University of Saugar, Saugar (M.P.), India