A NOTE ON THE BOCKSTEIN OPERATOR

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1. **Introduction.** The Stiefel-Whitney classes $w_1, \cdots, w_n$ of an n-plane bundle ξ over a space X are certain well-defined elements of $H^*(X; Z_2)$ (singular cohomology). For the effect of the Steenrod squares on them one has the Wu formulae $Sq^i w_j = \sum_0^i \binom{j-i+t-1}{t} w_{i-t} w_{j+t}$ [2]. In particular, since $Sq^1 = \beta_2$ (the Bockstein operator for the coefficient sequence $0 \to Z_2 \to Z_4 \to Z_2 \to 0$) one has

$$\text{(I)} \qquad \beta_2 w_{2i} = w_{2i+1} + w_1 w_{2i} \qquad (\text{here } w_j = 0 \text{ for } j > n).$$

The classes w_{2i+1}, for $2i+1 \leqq n$, are Z_2-reductions of classes that we shall write as $\tilde{w}_{2i+1}$, defined over a local coefficient system $\mathcal{Z}$ whose group is Z (the integers), the local system being determined by the orientation of the fiber, [1, p. 195] (incidentally, this also holds for w_n, in case of even n). With $\tilde{\beta}$ denoting the Bockstein operator for the sequence $0 \to \mathcal{Z} \to \mathcal{Z} \to Z_2 \to 0$, one has then, according to [1], the formulae

$$\tilde{\beta} w_{2i} = \tilde{w}_{2i+1} \qquad (\text{again } \tilde{w}_j = 0 \text{ for } j > n),$$

and consequently, writing ρ_2 for reduction mod 2 and $\tilde{\beta}_2$ for $\rho_2 \circ \tilde{\beta}$,

$$\text{(II)} \qquad \tilde{\beta}_2 w_{2i} = w_{2i+1}, \qquad \text{for } 0 < 2i \leqq n;$$

for $i = 0$ the situation is a little different, cf. §4 below.

Our aim is to connect the two formulae (I) and (II), by means of the theorem in §3.

2. **Pairing of coefficient sequences.** Let $0 \to A' \to A \to A'' \to 0$, $0 \to B' \to B \to B'' \to 0$ be two exact sequences of coefficient groups (or local systems for cohomology in X); let α, β be the corresponding Bockstein maps. There is the usual exact sequence $0 \to D \to A \otimes B \to A'' \otimes B'' \to 0$, where D is the natural image of $A' \otimes B \oplus A \otimes B'$ in $A \otimes B$. We can "divide" by the natural image of $A' \otimes B'$ in $A \otimes B$, and get the exact sequence (defining C as image of D)

$$(1) \qquad 0 \to C \to A \otimes B/i(A' \otimes B') \to A'' \otimes B'' \to 0,$$

which we call the product sequence; let γ be the associated Bockstein

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operator. Let $k: A'\otimes B' \to A'\otimes B \oplus A\otimes B'$ be defined by $k(a'\otimes b') = (a'\otimes b', \ -a'\otimes b')$. One verifies that the maps $A\to A''$, $B\to B''$ give rise to the following commutative diagram:

$$\begin{tikzcd} A'\otimes B\oplus A\otimes B' \arrow[r, "\lambda"] \arrow[dr, "\sigma"'] & (A'\otimes B\oplus A\otimes B')/k(A'\otimes B') \arrow[r, "\mu"] \arrow[d] & A'\otimes B''\oplus A''\otimes B' \arrow[d, "\nu"] \\ & D \arrow[r, "\tau"] & C \end{tikzcd}$$

Let now $u\in H^s(X;\ A'')$, $v\in H^t(X;\ B'')$ be given. We regard $\alpha u\cdot v+(-1)^s u\cdot\beta v$ as lying in $H^{s+t+1}(X;\ A'\otimes B''\oplus A''\otimes B')$, and $u\cdot v$ as lying in $H^{s+t}(X;\ A''\otimes B'')$. We write λ, resp. λ_*, for the coefficient maps, defined by λ, on cochains, resp. cohomology.

PROPOSITION. (a) $\gamma(u\cdot v) = \nu_*(\alpha u\cdot v + (-1)^s u\cdot\beta v)$; (b) $\alpha u\cdot v +(1-)^s u\cdot\beta v\in \operatorname{im}\mu_*$.

PROOF. Take cochains x, y representing u, v; "pull back" to cochains $\bar{x}$, $\bar{y}$ with coefficients in A, B. Then $d(\bar{x}\cdot\bar{y})$, with coefficients in $A\otimes B$, pulls back to D; similarly $d\bar{x}\cdot\bar{y}+(-1)^s\bar{x}\cdot d\bar{y}$ has coefficients in $A'\otimes B\oplus A\otimes B'$; their τ_*-, resp. $\tau_*\circ\sigma_*$-images are equal and represent $\gamma(u\cdot v)$. One checks that $\lambda_*(d\bar{x}\cdot\bar{y}+(-1)^s\bar{x}\cdot d\bar{y})$ is a cocycle (the two $A'\otimes B'$ have become equal). Applying μ_* to its cohomology class, we get part (b) of the proposition, and continuing with ν_* we get part (a).

3. **The twisted Bockstein operator.** Suppose X is pathwise connected and has a base point. Let p be any prime, and let w be an element, $\neq 0$, of $H^1(X;\ Z_p)$. By the universal coefficient theorem, w defines and is in fact equivalent to a homomorphism of $H_1(X)$ (over Z) onto Z_p. Composing with the Hurewicz map we get a homomorphism, w, of $\pi_1(X)$ onto Z_p. Now the group Z_{p^2} has an automorphism of order p, sending 1 into $p+1$, that leaves the subgroup Z_p and the corresponding quotient group elementwise fixed. We regard this as an action of Z_p on the sequence $0\to Z_p\to Z_{p^2}\to Z_p\to 0$. Composing with w, we get a left action of $\pi_1(X)$ on this sequence, and thus a sequence $0\to Z_p\to Z_{p^2}\to Z_p\to 0$ of local systems of coefficients in X, where the end terms are constant and the middle term has group Z_{p^2}. Write β^w for the Bockstein operator of this sequence; also write β_p for the Bockstein map of the constant sequence $0\to Z_p\to Z_{p^2}\to Z_p\to 0$. For $w=0$ define $\beta^0=\beta_p$.

THEOREM. *For any* $v\in H^*(X;\ Z_p)$ *we have*

$$\beta^w(v) = \beta_p(v) + w\cdot v.$$

PROOF. The case $w=0$ is trivial. Now let us take $w\neq 0$. First take $v=1$, the unit of $H^*(X; Z_p)$. Here the result to be proved is $\beta^w(1)=w$, since $\beta_p(1)=0$. It is sufficient to consider 1-simplices with both end points at the base point (use the 1st Eilenberg subcomplex of the singular complex). But then the relation to be proved is practically a tautology; note $(p+1)^i \equiv p\cdot i+1 \pmod{p^2}$.

Second, we apply the considerations of §2 to the sequence $0\to Z_p \to Z_{p^2}\to Z_p\to 0$ above and the constant sequence $0\to Z_p\to Z_{p^2}\to Z_p\to 0$. The general formulae simplify a good deal: $i(A'\otimes B')$ is 0, so that $D=C$; $k(A'\otimes B')$ is also 0; the product sequence is isomorphic to the first given sequence; so that $\gamma=\alpha(=\beta^w)$; both terms $A'\otimes B''$ and $A''\otimes B'$ are of the form $Z_p\otimes Z_p\approx Z_p$ and map onto $C\approx Z_p$ under ν. Part (a) of the proposition can then be interpreted as saying

$$\beta^w(u\cdot v) = \beta^w u\cdot v + (-1)^s u\cdot \beta_p v,$$

where all coefficients are Z_p and the product is the usual $\cup$-product over Z_p.

We now take $u=1$ and get $\beta^w(v)=\beta^w(1\cdot v)=\beta^w(1)\cdot v+\beta_p v=w\cdot v +\beta_p v$, thus proving the theorem.

As an example, with $p=2$, take infinite real projective space with $H^*(PR^\infty; Z_2)=Z_2[w]$. One finds that β^w vanishes on the odd powers of w, and β_2 on the even powers.

We now apply our theorem to the situation of §1, with $p=2$. We have of course $\tilde{\beta}_2=\beta^{w_1}$, and so (I) and (II) are equivalent by the theorem (note $+=-$ here).

4. **The case $i=0$.** For the Stiefel-Whitney classes in dimensions 0 and 1, Steenrod has the formula

$$\tilde{\beta}1 = \tilde{w}_1 \tag{I$_0$}$$

where one has to interpret 1 as the integral unit class, and one uses for the construction of $\tilde{\beta}$ the sequence $0\to Z\to \mathfrak{B}\to Z\to 0$, where the group of $\mathfrak{B}$ is $Z\oplus Z$, and $\pi_1(X)$ acts by interchanging the two terms. Reducing mod 2 we get the sequence $0\to Z_2\to \mathfrak{V}\to Z_2\to 0$, where the group of $\mathfrak{V}$ is $Z_2\oplus Z_2$, and the relation

$$\rho_2\circ\tilde{\beta}(1) = w_1, \tag{II$_0$}$$

where now 1 is over Z_2. To "explain" this we consider the situation analogous to that of §3: $w_1\neq 0$, in $H^1(X; Z_2)$; $\pi_1(X)$ acting on $Z_2\oplus Z_2$ "according to w"; β' the corresponding Bockstein operator. One verifies as in §3 again $\beta'(1)=w$; and $\rho_2\circ\tilde{\beta}$ in (II$_0$) is just the β' for $w=w_1$.

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