A RING PRIMITIVE ON THE RIGHT BUT NOT ON THE LEFT

G. M. BERGMAN

Jacobson [1, p. 4] writes, "It is not known whether [right] primitivity implies left primitivity. It seems unlikely that it does, but no examples of [right] primitive rings which are not left primitive are known." Such an example is here constructed.

Let D be a division ring, and $\alpha: D \to D$ an endomorphism. Let A be the ring of formal polynomials $\sum_{i \geq 0} d_i Y^i$ ($d_i \in D$, nonzero for only finitely many i) with multiplication determined by the rule $Yd = \alpha(d)Y$. Such rings are without zero divisors, and every left ideal of one is principal. The proofs are exactly as for the ordinary commutative rings of polynomials, cf. [2, p. 483].

Now let $D = \mathbf{Q}(X)$, where $\mathbf{Q}$ denotes the field of rationals, and let α be the map $r(X) \to r(X^2)$. In this case, we have the following result.

PROPOSITION 1. *Any subring $B \subset A$ containing X and Y is right primitive.*

PROOF. We observe that for any $r \in \mathbf{Q}(X)$ there is a unique $r^* \in \mathbf{Q}(X)$ such that $(r(X) + r(-X))/2 = r^*(X^2)$. We use this in defining the structure of a right A-module on $\mathbf{Q}(X)$ such that, if r and s are elements of $\mathbf{Q}(X)$, $r \cdot s = rs$ and $r \cdot Y = r^*$. To see that this is actually a right A-module structure, it suffices to verify that $(r \cdot Y) \cdot s = (r \cdot \alpha(s)) \cdot Y$, i.e., that $r^* s = (r\alpha(s))^*$. Now we have $(r^*s)(X^2) = \frac{1}{2}(r(X) + r(-X))s(X^2) = \frac{1}{2}(r(X)\alpha(s)(X) + r(-X)\alpha(s)(-X)) = (r\alpha(s))^*(X^2)$, whence $r^*s = (r\alpha(s))^*$.

We observe:

$$X^n \cdot Y^m = \begin{cases} X^{n/2^m} & \text{if } 2^m \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Let M be the B-submodule of $\mathbf{Q}(X)$ that is generated by X. We claim that M is irreducible. For let us be given a nonzero element $P(X)/Q(X) \in M$, where $P(X)$ and $Q(X)$ are polynomials. Take $2^a > \deg P$. Let $c = 1/(\text{leading coefficient of } P)$. Then $[P(X)/Q(X)] \cdot cQ(X)X^{2^a - \deg P}\, Y^a = [cP(X)X^{2^a - \deg P}] \cdot Y^a$. The element in square brackets is a polynomial with leading term X^{2^a}, and constant term zero. Hence Y^a, applied to it, gives X, which in turn generates M.

Next we show that M is faithful: Let $0 \neq b = \sum r_i(X) Y^i \in B$. Choose

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a polynomial $P(X)$ such that the $R_i(X)=P(X)r_i(X)$ are all polynomials. We shall find $m\in M$ such that $m\cdot \sum R_i(X)Y^i\neq 0$, which will give us $mP(X)\cdot b\neq 0$.

Let j be the least integer such that $R_j\neq 0$. Let

$$d = \max_i (\deg R_i - \deg R_j).$$

Choose $n>0$ so that $2^n\geqq \deg R_j$, $2^{n-j-1}\geqq d$. Consider $X^{2^n-\deg R_j}\cdot\sum R_i(X)Y^i$. The exponent of X in the highest-power term of $X^{2^n-\deg R_j}\cdot R_i(X)Y^i$ is $(2^n - \deg R_j + \deg R_i)/2^i$, or less if this is not integral. For $i=j$ this comes to 2^{n-j}. For $i>j$, it is $\leqq(2^n+d)/2^i <2^{n-j-1}+d\leqq 2^{n-j-1}+2^{n-j-1}=2^{n-j-1}$. So only the jth term contributes to the coefficient of $X^{2^{n-j}}$, which is thus nonzero. Q.E.D.

Now, for every odd integer $n>0$, let v_n be the valuation on $\boldsymbol{Q}(X)$ induced by the nth cyclotomic polynomial. For $r\in\boldsymbol{Q}(X)$, we have $v_n(r(X^2))=v_n(r(X))$.[1]

Suppose, in the more general context of the second paragraph, that a valuation v on D satisfies $v(\alpha(d))=v(d)$ for all $d\in D$. Then it follows immediately that v is extended to a valuation on A by the definition $v(\sum d_iY^i)=\min_i v(d_i)$.

Suppose we have an infinite set V of valuations of this sort, with the properties that at any $d\in D-\{0\}$, only finitely many of them are nonzero, and that for each $v\in V$, there is an $x_v\in D$ on which all valuations are nonnegative, and v is positive. Let us designate by B the intersection of the valuation subrings of the valuations in V, i.e., the subring consisting of those elements on which all the valuations are nonnegative.

In the specific case in question, B contains X and Y, and so, by Proposition 1, is right primitive. But we shall now show that, in general, it cannot be left primitive.

PROPOSITION 2. *B is not left primitive.*

PROOF. Let I be any left ideal of B. We shall show that either it is not maximal or the annihilator in B of B/I is not zero.

AI, being a left ideal of A, is principal. Let g be a generator. We can assume its leading coefficient is 1.

Case 1. g has Y-degree $d>0$.

General observation: given any $v\in V$ and nonzero $a=\sum d_iY^i\in A$,

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[1] One of many ways to see this is as follows: Consider $r(X)$ as a meromorphic function on the complex plane. It is clear that the order of $r(X^2)$ at z_0 is exactly the order of r at z_0^2 if $z_0\neq 0$. But if z_0 is a primitive nth root of unity for odd n, so is z_0^2, and the orders of r at z_0 and z_0^2 are the same. Hence the orders of $r(X)$ and $r(X^2)$ at z_0 are the same.

there must be some i such that $v(a) = v(d_i)$. If i_0 is the greatest such, we shall say that a is of relativized v-degree i_0. It is clear that the relativized v-degree of aa' is the sum of the relativized v-degrees of a and a'.

Choose v such that $v(g) = 0$. Then g has relativized v-degree d. Hence any nonzero element of AI has relativized v-degree >0.

Now $x_v \notin I$. Hence, if I were maximal, we could write $bx_v + i = 1$, $b \in B$, $i \in I$. But then we would have $i = 1 - bx_v$, which has relativized v-degree 0; contradiction. Hence I is not maximal.

Case 2. $g = 1$.

General observation: given nonzero $d \in D$, an element of A can be written in the form bd with $b \in B$ if and only if it can be written db' with $b' = d^{-1}bd \in B$, since all valuations of b and $d^{-1}bd$ are the same. Hence $Bd = dB$.

Now let $1 = \sum a_k i_k$, $a_k \in A$, $i_k \in I$. For a sufficiently large (but finite!) product x of the x_v's, $xa_k \in B$ for all k. Hence $x = \sum (xa_k) i_k \in I$.

Hence $I \supset Bx = xB$, and so the module B/I is not faithful. Q.E.D.

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References

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University of California, Berkeley