CHARACTERIZATION OF REGULAR HAUSDORFF MOMENT SEQUENCES

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The theorem of this paper is suggested to the author by H. S. Shapiro's proposal [**3**], to the effect that

$$\text{(1)} \qquad \operatorname*{Lim}_{n\to\infty} \sum_{k=0}^{n} (-1)^k f(k/n) \binom{n}{k} \Big/ 2^n = 0$$

for each f in $C[0, 1]$, the class of continuous functions from $[0, 1]$ to the complex numbers. Indeed, we find that this proposal is precisely the assertion that *the sequence* μ, $\mu_n = 1/2^n$ $(n = 0, 1, \cdots)$, *is a regular Hausdorff moment sequence with limit* 0. The latter assertion is well known: for relevant ideas, one may consult [**4**, Chapters 14 and 16] and, especially, [**4**, p. 309] regarding the Euler-Knopp mean $(E, 1/2)$ associated with the aforementioned sequence μ. The connecting link is the following result.

THEOREM. *In order that the infinite complex number sequence* μ, *with* $\mu_0 = 1$, *be a regular Hausdorff moment sequence, it is necessary and sufficient that, for each* f *in* $C[0, 1]$, *the limit*

$$\text{(2)} \qquad L(f) = \operatorname*{Lim}_{n\to\infty} \sum_{k=0}^{n} (-1)^{n-k} f(k/n) \binom{n}{k} \Delta^{n-k}\mu_k$$

exist, and in this case, for each f *in* $C[0, 1]$, $L(f) = f(1) \cdot \mathrm{Lim}_{n\to\infty}\, \mu_n$.

INDICATION OF PROOF. If $L(f)$ exists for each f in $C[0, 1]$, then there is a number B such that

$$\text{(3)} \qquad \sum_{k=0}^{n} \binom{n}{k} \left| \Delta^{n-k}\mu_k \right| \leqq B \qquad (n = 0, 1, \cdots).$$

This is an application of the "principle of uniform boundedness" in the linear space $C[0, 1]$, normed as usual by $\|f\| = \text{L.U.B. } |f(x)|$: see [**1**, Chapter 2], especially the remarks on pp. 80–81 thereof. By a theorem due to F. Riesz [**4**, p. 271], (3) is equivalent to the existence of a complex-valued function ϕ, of bounded variation on $[0, 1]$, such that

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$$\phi(0) = 0 \quad \text{and} \quad \mu_n = \int_0^1 I^n d\phi \qquad (n = 0, 1, \cdots), \tag{4}$$

where I denotes the identity function on $[0, 1]$. In this circumstance, the sequence μ has the limit $\phi(1)-\phi(1-)$, and there is a theorem due to F. Hausdorff [**4**, p. 309] to the effect that μ is *regular* only in case ϕ is continuous at 0, *i.e.*, if and only if $\phi(0+)=0$.

From the preceding considerations, together with the Weierstrass theorem on uniform approximation by polynomials, we see that the theorem will be established once we have the following lemma.

LEMMA. *If μ is the Hausdorff moment sequence given by* (4) *with ϕ of bounded variation then, for each polynomial g, the sequence $\lambda(g)$,*

$$\lambda_n(g) = \sum_{k=0}^{n} \left\{(-1)^{n-k} g(k/n) \binom{n}{k} \Delta^{n-k}\mu_k\right\} - (-1)^n g(0)\phi(0+), \tag{5}$$

converges and has the limit $g(1)[\phi(1)-\phi(1-)]$.

PROOF OF LEMMA. Starting with a positive integer n, and the binomial theorem in the form

$$h_0(z) = \sum_{k=0}^{n} z^k \binom{n}{k} (1 - I)^{n-k} I^k = (1 - I + zI)^n, \tag{6}$$

one establishes inductively the fact that if p is a positive integer not greater than n then $h_p(z)=zh'_{p-1}(z)$ is given by

$$\sum_{k=0}^{n} z^k k^p \binom{n}{k} (1 - I)^{n-k} I^k = \sum_{q=1}^{p} \binom{n}{q} d_{p,q} (1 - I + zI)^{n-q} (zI)^q, \tag{7}$$

where $d_{p,1}=1$, $d_{p,p}=p!$, and $d_{p+1,q}=(d_{p,q-1}+d_{p,q})q$ if $1<q<p$: see [**2**, p. 57] for a similar use of the numbers $d_{p,q}$. Now by taking z to be -1 in (7), and multiplying by $(-1)^n$, we see that

$$\sum_{k=0}^{n} (-1)^{n-k} k^p \binom{n}{k} (1 - I)^{n-k} I^k = \sum_{q=1}^{p} \binom{n}{q} d_{p,q} (2I - 1)^{n-q} I^q. \tag{8}$$

From (6), we see that for each positive integer n

$$\lambda_n(I^0) = \int_{0+}^{1-} (2I - 1)^n d\phi + \phi(1) - \phi(1-), \tag{9}$$

which clearly has limit $\phi(1)-\phi(1-)$ as $n\to\infty$. On the other hand. we see from (8) that, for $0<p<n$,

$$\lambda_n(I^p) = \sum_{q=1}^{p} (1/n)^p \binom{n}{q} d_{p,q} \int_0^1 (2I - 1)^{n-q} I^q d\phi, \tag{10}$$

which also has limit $\phi(1) - \phi(1-)$ as $n \to \infty$. This completes the proof.

The theorem, now established, seems a natural companion to the following proposition, which is easily established by application of the "principle of uniform boundedness" and computation with Bernstein polynomials, and which we state without proof:

PROPOSITION. *In order that the infinite complex number sequence μ be a Hausdorff moment sequence, it is necessary and sufficient that, for each f in $C[0, 1]$, the limit*

$$M(f) = \operatorname*{Lim}_{n \to \infty} \sum_{k=0}^{n} f(k/n) \binom{n}{k} \Delta^{n-k} \mu_k$$

exist, and in this case, for each f in $C[0, 1]$, $M(f) = \int_0^1 f\, d\phi$ where ϕ is a function of bounded variation having μ as its moment sequence.

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