VON NEUMANN'S THEOREM ON ABELIAN FAMILIES OF OPERATORS

B. R. GELBAUM

The theorem referred to in the title is the following: If $\{A_n\}$ is a countable family of bounded commuting normal operators over an arbitrary Hilbert space (not necessarily separable), then there is a resolution of the identity $\{E(t) \mid 0 \leqq t \leqq 1\}$ and a sequence of continuous functions $\{a_n(t)\}$ such that, for all n,

$$A_n = \int_0^1 a_n(t) dE(t).$$

The short proof below resembles but differs from von Neumann's original proof [**2**; **3**].

Let $\mathfrak{A}$ be the uniformly closed algebra generated by the set $\{I, A_n, A_m^*, n, m = 1, 2, \cdots\}$. Then, from the general theory of Banach algebras [**1**] we see that $\mathfrak{A}$ and $C(\mathfrak{M})$ are isometrically isomorphic and that the maximal ideal space $\mathfrak{M}$ of $\mathfrak{A}$ is a compact metric space, since $\mathfrak{A}$ is separable. Hence there is a mapping $f: S \to \mathfrak{M}$ of the Cantor set S onto $\mathfrak{M}$. For t in $[0, 1]$, let $\hat{E}_t(M)$ be the characteristic function of the set $f([0, t) \cap S) \subset \mathfrak{M}$. Each of these sets, as the union of the compact sets $f([0, t - 1/n] \cap S)$ is a Borel, hence since $\mathfrak{M}$ is metric, a Baire set.

In accordance with the isometric isomorphism between the set of bounded Baire functions on $\mathfrak{M}$ and a super-ring of $\mathfrak{A}$ [**1**, 26 F, 26 G], $\hat{E}_t(M)$ corresponds to a projection $E(t)$, and clearly $\{E(t) \mid 0 \leqq t \leqq 1\}$ is a resolution of the identity.

For B in $\mathfrak{A}$, let $b(t)$ be defined as follows:

$$b(t) = \begin{cases} \hat{B}(f(t)), & t \in S; \\ \alpha b(t_1) + \beta b(t_2), & t = \alpha t_1 + \beta t_2, \text{ where } (t_1, t_2) \text{ is one of the intervals} \\ & \quad \text{deleted in forming } S \text{ and } 0 \leqq \alpha, \beta;\ \alpha + \beta = 1. \end{cases}$$

Since f and $\hat{B}$ are continuous, $b(t)$ is continuous. A direct computation shows $B = \int_0^1 b(t) dE(t)$ and, in particular, $A_n = \int_0^1 a_n(t) dE(t)$.

Indeed, for $\epsilon > 0$, there is a $\delta > 0$ such that if $|t_1 - t_2| < \delta$ then $|b(t_1) - b(t_2)| < \epsilon$. Thus let $0 = t_0 < t_1 < \cdots < t_n = 1$ where $\max_i |t_{i+1} - t_i| < \delta$. For $\tau_i \in [t_i, t_{i+1})$ we find

Received by the editors December 24, 1962 and, in revised form, February 13, 1963.

$$\left\| B - \sum_{i=0}^{n-1} b(\tau_i)[E(t_{i+1}) - E(t_i)] \right\|$$
$$= \left\| \hat{B}(M) - \sum_{i=0}^{n-1} b(\tau_i)[\hat{E}_{t_{i+1}}(M) - \hat{E}_{t_i}(M)] \right\|_\infty .$$

For any $M \in \mathfrak{M}$, there is a unique i_0 such that

$$M \in f([0, t_{i_0+1}) \cap S) \backslash f([0, t_{i_0}) \cap S).$$

For this M, then, $\hat{E}_{t_{i+1}}(M) - \hat{E}_{t_i}(M) = 0$ unless $i = i_0$, in which case $\hat{E}_{t_{i+1}}(M) - \hat{E}_{t_i}(M) = 1$. Furthermore, $M = f(\tau)$, where $\tau \in (t_{i_0}, t_{i_0+1}) \cap S$. Thus $\hat{B}(M) = b(\tau)$ and

$$|\hat{B}(M) - b(\tau_{i_0})| = |b(\tau) - b(\tau_{i_0})| < \epsilon.$$

In short, $\| \hat{B}(M) - \sum_{i=0}^{n-2} b(\tau_i)[\hat{E}_{t_{i+1}}(M) - \hat{E}_{t_i}(M)]\|_\infty < \epsilon$, and finally $\|B - \sum_{i=0}^{n-1} b(\tau_i)[E(t_{i+1}) - E(t_i)]\| < \epsilon$. The required conclusion then follows.

The usual extensions of the above theorem to the cases where (a) the A_n are not necessarily bounded or (b) $\{A_n\}$ is replaced by a not necessarily countable family $\{A_\lambda\}$ of not necessarily bounded, commuting normal operators on a *separable* Hilbert space, follow readily [2].

On the other hand, let $\mathfrak{a}$ be a cardinal greater than $2^{2^{\aleph_0}}$, and let $\Lambda = \{\lambda\}$ be a set of cardinality $\mathfrak{a}$. Then the set $l_2(\Lambda) \equiv \{x(\lambda) \mid x(\lambda)$ complex-valued, $\sum_{\lambda \in \Lambda} |x(\lambda)|^2 < \infty\}$ is a (highly nonseparable) Hilbert space on which the projections $P_\mu: x(\lambda) \rightarrow y(\lambda) = x(\mu)\delta_{\mu\lambda}$, form a commuting family of bounded Hermitian operators. If there were some resolution of the identity $\{E(t)\}$ such that for $\mu \in \Lambda$, $P_\mu = \int_0^1 p_\mu(t) dE(t)$, where $p_\mu(t)$ is a complex-valued function, then the cardinality of the set $\{p_\mu(t)\}$ would be $\mathfrak{a} > 2^{2^{\aleph_0}}$, which is impossible, since the cardinality of the set of *all* complex-valued functions on $[0, 1]$ is $2^{2^{\aleph_0}}$.

Bibliography

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University of Minnesota