

BRACKET AND EXPONENTIAL FOR A NEW TYPE OF VECTOR FIELD

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In [2] Robert Hermann introduced the concept of tangent vector fields on the space of functions from one manifold to another. He applied these to give a new proof of the Cartan-Kähler theorem. An example of such vector fields are maps from the jet space to the tangent bundle of the target space which commute with projections. It is this class of vector fields which we study here.

Using prolongations a Lie bracket operation is defined and justified on the grounds that it agrees with the primitive definition when the latter has meaning here. By similar methods an exponential expansion is deduced. An example is given which shows that the 1-parameter transformation groups on the function space cannot be considered a parameter space for a pseudo group in Kuranishi's sense [3], for it need not involve infinite analytic mappings.

1. Introduction. Every mapping and manifold will be smooth of class C^∞ unless otherwise noted. If N and M are two manifolds, $J^k = J^k(N, M)$ is the manifold of k -jets $j_x^k(f)$ of order k of maps $f: N \rightarrow M$ (see [1]). α and β denote the customary source and target projections. $T(M)$ is the tangent bundle of M , M_y the tangent space at $y \in M$. $\pi: T(M) \rightarrow M$ is the bundle projection. $C^\infty(N, M)$ is the set of all C^∞ maps on N into M .

DEFINITION 1. A k -vector field on $C^\infty(N, M)$ is a map $\theta: J^k \rightarrow T(M)$ such that $\pi \circ \theta = \beta$.

Hermann studied k -vector fields as a special class of "formal tangent vector" fields [2, p. 8]. If $f: N \rightarrow M$, a "vector" along f is a map $\psi: N \rightarrow T(M)$ with $\psi(x) \in M_{f(x)}$ for all $x \in N$. This is what one would get as the derivative of a 1-parameter family $f_t \in C^\infty(N, M)$ where $f_0 = f$. Each k -vector field θ defines a vector along f by $\psi(x) = \theta(j_x^k(f))$.

Let $I = (-\epsilon, \epsilon)$. An *integral curve* of θ starting at $f_0 \in C^\infty(N, M)$ is a 1-parameter family $f: N \times I \rightarrow M$ with $f(x, 0) = f_0(x)$ and

$$\frac{\partial f}{\partial t}(x, t) = \theta(j_x^k(f)).^1$$

Presented to the Society, January 27, 1963; received by the editors September 15, 1962 and, in revised form, January 7, 1963.

¹ The referee observed that this is a special system of the form $\partial f / \partial t = F(\partial I^k / \partial x^k, \dots)$. Hence even in the analytic case, solutions need not exist (for $k > 1$) with arbitrary initial conditions.

In coordinates this is seen to be a Cauchy-Kowalewski system of order k . By uniqueness of C^ω (i.e., analytic) solutions we see that in the C^ω case if $g_0(x) = f(x, t)$, and $g(x, \tau)$ is an integral curve starting at g_0 , then $g(x, \tau) = f(x, t + \tau)$. Thus, these integral curves, when they exist and are unique, behave as the orbits of a local 1-parameter group.

2. An example. Let $N = M = E^1$, Euclidean 1-space. Let (x) , (y) , and (x, y, p) be coordinates on N , M , and $J^1(N, M)$, respectively. Consider the 1-vector field $\theta(x, y, p) = p(\partial/\partial y)_y$. Given $f_0(x): N \rightarrow M$, $f(x, t)$ must satisfy

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}, \quad f(x, 0) = f_0(x).$$

We think of $f(x, t)$ as the image of f_0 under a transformation F_t on functions: $f(x, t) = F_t(f_0)$.

To compare the action of F_t with M. Kuranishi's concept of infinite analytic mappings [3], consider all convergent power series at the origin: $f_0 = \sum a_n x^n$. Then

$$F_t(f_0) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} a_{n+m} \frac{(n+m)!}{n!} \right) x^n$$

is not an infinite analytic mapping in Kuranishi's sense because the coefficient of x^n in $F_t(f_0)$ is an infinite series in the coefficients of f_0 rather than a polynomial.

3. Lie bracket. If θ is a k -vector field let $h_t: J^k \rightarrow M$ for $-\epsilon < t < \epsilon$ satisfy $h_0 = \beta$ and $(\partial h_t / \partial t)_{t=0} = \theta$.² If $f \in C^\infty(N, M)$, the k th prolongation of f , $p^k(f) \in C^\infty(N, J^k)$ is defined by $p^k(f)(x) = j_x^k(f)$. Similarly, the r th prolongation of h_t , $p^r(h_t) \in C^\infty(J^{k+r}, J^r)$, is defined by

$$p^r(h_t)(j_x^{r+k}(f)) = j_x^r(h_t \circ p^k(f)).$$

DEFINITION 2. $P^r(\theta)$, the r th prolongation of θ , is defined to be

$$P^r(\theta)(j_x^{r+k}(f)) = \frac{\partial}{\partial t} [p^r(h_t)(j_x^{r+k}(f))]_{t=0}.$$

Then $P^r(\theta): J^{r+k} \rightarrow T(J^r)$, and if π denotes the projection of $T(J^r)$ onto J^r and ρ_r^{r+k} carries $j_x^{r+k}(f)$ in J^{r+k} to $j_x^r(f)$ in J^r , then $\pi \circ P^r(\theta) = \rho_r^{r+k}$.

Kuranishi's notion of formal partial derivative is very useful in describing $P^r(\theta)$ in coordinates. If (x^1, \dots, x^n) and (y^1, \dots, y^m)

² That is, if $F: M \rightarrow R$, $\theta(F)(j_x^k(f)) = d[F \circ h_t(j_x^k(f))]/dt|_{t=0}$.

are local coordinates on $U \subset N$ and $V \subset M$, respectively, let $(x^i, y^\lambda, p_{j_1}^\lambda, \dots, p_{j_1 \dots j_k}^\lambda)$ be local coordinates on $\alpha^{-1}(U) \cap \beta^{-1}(V)$, where $\lambda = 1, \dots, m; i_k = 1, \dots, n$. If $u: J^k \rightarrow R$ (real numbers), define $\partial_j^\# u: J^{k+1} \rightarrow R$ to be

$$\partial_j^\# u(j_x^{k+1}(f)) = \frac{\partial u}{\partial x^i} + \frac{\partial u}{\partial y^\lambda} p_j^\lambda + \dots + \frac{\partial u}{\partial p_{i_1 \dots i_k}^\lambda} p_{j_1 \dots j_k}^\lambda.$$

This operator is linear and has the important property that if $f: U \rightarrow V$, then $\partial(u \circ p^k(f))/\partial x^i = \partial_j^\# u(p^{k+1}(f))$. Using these facts and Definition 2, it is possible to prove the following

LEMMA 1. If on $\alpha^{-1}(U) \cap \beta^{-1}(V)$, $\theta = \theta^\lambda (\partial/\partial y^\lambda)$, then

$$P^r \theta = \theta^\lambda \frac{\partial}{\partial y^\lambda} + \partial_{j_1}^\# \theta^\lambda \frac{\partial}{\partial p_{i_1}^\lambda} + \dots + \partial_{j_1}^\# (\dots \partial_{j_r}^\# (\theta^\lambda)) \frac{\partial}{\partial p_{i_1 \dots i_r}^\lambda}.$$

Moreover, P^r is linear.

DEFINITION 3. Let θ and ψ be r - and s -vector fields, respectively. Then the *Lie bracket* of θ and ψ is

$$[\theta, \psi] = P^s \theta \circ \psi - P^r \psi \circ \theta.$$

By the composition notation we mean to interpret ψ as an operator carrying $C^\infty(M, R)$ into $C^\infty(J^s, R)$. Similarly, $P^s \theta$ carries $C^\infty(J^s, R)$ into $C^\infty(J^{r+s}, R)$.

Alternatively, if $h_t: J^r \rightarrow M$ and $g_\tau: J^s \rightarrow M$ satisfy $h_0 = \beta$, $g_0 = \beta$, $-\epsilon < t < \epsilon$, $-\epsilon < \tau < \epsilon$, and $(\partial h_t / \partial t)_{t=0} = \theta$, $(\partial g_\tau / \partial \tau)_{\tau=0} = \psi$, then

$$(P^s \theta \circ \psi)(j_x^{r+s}(f)) = \frac{\partial^2}{\partial t \partial \tau} [g_\tau \circ p^s h_t(j_x^{r+s}(f))]_{\tau=t=0}.$$

This representation is convenient when proving

LEMMA 2. $[\theta, \psi]$ is an $(r+s)$ -vector field.

LEMMA 3. $P^q(P^s \theta \circ \psi) = P^{s+q} \theta \circ P^q \psi$.

PROOF. This follows from $p^q(g_\tau \circ p^s h_t) = p^q g_\tau \circ p^{s+q} h_t$, which is a consequence of the definitions.

LEMMA 4 (LIE IDENTITY).

$$[[\theta, \psi], \phi] + [[\psi, \phi], \theta] + [[\phi, \theta], \psi] = 0.$$

PROOF. Suppose θ, ψ, ϕ are r -, s -, and q -vector fields, respectively. The left side of the above equation is

$$\begin{aligned}
& P^a(P^s\theta \circ \psi - P^r\psi \circ \theta) \circ \phi - P^{s+r}\phi \circ (P^s\theta \circ \psi - P^r\psi \circ \theta) \\
& + P^r(P^a\psi \circ \phi - P^s\phi \circ \psi) \circ \theta - P^{a+s}\theta \circ (P^a\psi \circ \phi - P^s\phi \circ \psi) \\
& + P^s(P^r\phi \circ \theta - P^a\theta \circ \phi) \circ \psi - P^{r+a}\psi \circ (P^r\phi \circ \theta - P^a\theta \circ \phi).
\end{aligned}$$

This equals zero by Lemma 3 above.

It follows from Definition 3 of the Lie bracket that $[\theta, \theta] = 0$. Hence we have proved

THEOREM 1. *If V^k is the linear space of all k -vector fields, $k = 0, 1, \dots$, then the direct sum*

$$V = \sum_{k=0}^{\infty} \oplus V^k$$

is a graded Lie algebra under the Lie bracket.

A definition of Lie bracket in the older literature used local 1-parameter transformation groups. If θ and ψ are vector fields on a manifold generating local transformation groups Θ_t and Ψ_τ , respectively, then $[\theta, \psi]$ is the vector field obtained by transforming Θ_t by Ψ_τ . That is,

$$[\theta, \psi] = \frac{\partial^2}{\partial t \partial \tau} [\Psi_\tau \circ \Theta_t \circ \Psi_{-\tau}]_{t=\tau=0}.$$

We do not have 1-parameter local groups on $C^\infty(N, M)$ in general. However, one may observe how individual functions behave when they belong to integral curves of θ and ψ .

THEOREM 2. *Let θ and ψ be r - and s -vector fields, respectively. Let $I = (-\epsilon, \epsilon)$. Let $f(x): N \rightarrow M$. Suppose:*

a. $\tilde{f}(x, t): N \times I \rightarrow M$ and satisfies

$$\frac{\partial \tilde{f}}{\partial t}(x, t) = \psi(j_x^r(\tilde{f})), \quad \tilde{f}(x, 0) = f(x);$$

b. $f^*(x, t, \tau): N \times I \times I \rightarrow M$ and satisfies

$$\frac{\partial f^*}{\partial t}(x, t, \tau) = \theta(j_x^s(f^*)), \quad \tilde{f}^*(x, t, 0) = \tilde{f}(x, t);$$

c. $\tilde{f}^*(x, t, \tau, t): N \times I \times I \times I \rightarrow M$ and satisfies

$$\frac{\partial \tilde{f}^*}{\partial t}(x, t, \tau, t) = \psi(j_x^r(\tilde{f}^*)), \quad \tilde{f}^*(x, t, \tau, 0) = f^*(x, t, \tau).$$

Then

$$\frac{\partial^2}{\partial t \partial \tau} \bar{f}^*(x, -t, \tau, t) \big|_{t=\tau=0} = [\theta, \psi](j_x^{s+r}(f)).$$

PROOF.

$$\begin{aligned} \frac{\partial^2}{\partial t \partial \tau} \bar{f}^*(x, -t, \tau, t) \big|_{t=\tau=0} &= -\frac{\partial^2}{\partial t \partial \tau} f^*(x, t, \tau, t) \big|_{t=\tau=0} + \frac{\partial^2}{\partial \tau \partial t} \bar{f}^*(x, t, \tau, t) \big|_{t=\tau=0} \\ &= -\frac{\partial^2}{\partial t \partial \tau} f^*(x, t, \tau) \big|_{t=\tau=0} + \frac{\partial}{\partial \tau} \psi[j_x^r(f^*)]_{t=\tau=0} - \frac{\partial}{\partial t} \theta(j_x^s(\bar{f})) \big|_{t=0} \\ &\quad + \frac{\partial}{\partial \tau} \psi[j_x^r(f^*(x, 0, \tau))]_{\tau=0}. \end{aligned}$$

We can calculate these using local coordinates (see above).

$$\begin{aligned} \frac{\partial}{\partial t} \theta^\lambda \left(x^i, \bar{f}^\mu, \frac{\partial \bar{f}^\mu}{\partial x^j}, \dots, \frac{\partial^s \bar{f}^\mu}{\partial x^{j_1} \dots \partial x^{j_s}} \right) &= \frac{\partial \theta^\lambda}{\partial y^\mu} \frac{\partial \bar{f}^\mu}{\partial t} + \frac{\partial \theta^\lambda}{\partial p_i^\mu} \frac{\partial^2 \bar{f}^\mu}{\partial t \partial x^j} + \dots + \frac{\partial \theta^\lambda}{\partial p_{j_1 \dots j_s}^\mu} \frac{\partial^{s+1} \bar{f}^\mu}{\partial t \partial x^{j_1} \dots \partial x^{j_s}} \\ &= \frac{\partial \theta^\lambda}{\partial y^\mu} \psi^\mu + \frac{\partial \theta^\lambda}{\partial p_i^\mu} \partial_j^\# \psi^\mu + \dots + \frac{\partial \theta^\lambda}{\partial p_{j_1 \dots j_s}^\mu} \partial_{j_1}^\# (\dots \partial_{j_s}^\#)(\psi^\mu) \\ &= P^* \psi \circ \theta. \end{aligned}$$

In the same way the last term in the first equation becomes $P^* \theta \circ \psi$. Q.E.D.

4. The exponential map.

LEMMA 5. Let θ and ψ be r - and s -vectors, respectively. Suppose $f(x, t): N \times I \rightarrow M$ satisfies $\theta(j_x^r(f)) = (\partial f / \partial t)(x)$. Then

$$\frac{\partial}{\partial t} \psi(j_x^s(f)) = P^* \theta \circ \psi(j_x^{r+s}(f)).$$

PROOF. In the previous local coordinates, we are given that $\theta^\lambda(j_x^r(f)) = (\partial f^\lambda / \partial t)(x, t)$. Hence

$$\begin{aligned} \frac{\partial}{\partial t} \psi^\lambda(j_x^s(f)) &= \frac{\partial \psi^\lambda}{\partial y^\mu} \frac{\partial f^\mu}{\partial t} + \frac{\partial \psi^\lambda}{\partial p_i^\mu} \frac{\partial^2 f^\mu}{\partial t \partial x^j} + \dots + \frac{\partial \psi^\lambda}{\partial p_{j_1 \dots j_s}^\mu} \frac{\partial^{s+1} f^\mu}{\partial t \partial x^{j_1} \dots \partial x^{j_s}} \\ &= \frac{\partial \psi^\lambda}{\partial y^\mu} \theta^\mu + \frac{\partial \psi^\lambda}{\partial p_i^\mu} \partial_j^\# \theta^\mu + \dots + \frac{\partial \psi^\lambda}{\partial p_{j_1 \dots j_s}^\mu} \partial_{j_1}^\# (\dots \partial_{j_s}^\#(\theta^\mu)), \end{aligned}$$

evaluated at $j_x^{s+r}(f)$. The result follows from Lemma 1.

THEOREM 3. *Let θ be an r -vector field, where $N = E^n$, $M = E^m$. Let $f(x, t): E^n \times I \rightarrow E^m$ satisfy $(\partial f / \partial t)(x, t) = \theta(j_x^r(f))$, where*

$$f(x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f_n(x)$$

converges on $E^n \times I$. Then identifying $T(E^m)$ with E^m ,

$$f_n(x) = P^{(n-1)r} \theta \circ P^{(n-2)r} \theta \circ \cdots \circ P^r \theta \circ \theta(j_x^{rn}(f(x, 0))).$$

PROOF. By repeated application of Lemma 5 and from $\partial f / \partial t = \theta(j_x^r(f))$ it follows that

$$\frac{\partial^2 f}{\partial t^2}(x, t) = P^r \theta \circ \theta(j_x^{2r}(f)),$$

$$\frac{\partial^3 f}{\partial t^3}(x, t) = P^{2r} \theta \circ P^r \theta \circ \theta(j_x^{3r}(f)), \cdots \text{ Q.E.D.}$$

Theorem 3 shows that in thinking of θ as an infinitesimal transformation on $C^\infty(N, M)$, one should consider

$$\exp(\theta)(f_0) = \sum_{n=0}^{\infty} \frac{t^n}{n!} P^{(n-1)r} \theta \circ \cdots \circ P^r \theta \circ \theta(j_x^{rn}(f_0))$$

as a generalized exponential formula.

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