OSCILLATION THEOREMS FOR ELLIPTIC EQUATIONS

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This paper deals with oscillatory behavior of solutions of singular self-adjoint elliptic equations of the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial u}{\partial x_i}\right) + cu = 0. \tag{1}$$

We shall assume that the coefficients of all equations of the form (1) satisfy

(i) $a_{ij}(x)$ differentiable, $c(x)$ continuous;

(ii) $a_{ij}=\bar{a}_{ji}$, $c=\bar{c}$;

(iii) $\sum a_{ij}\xi_i\bar{\xi}_j>0$ for all complex n-tuples $(\xi_1, \cdots, \xi_n)\neq(0, \cdots, 0)$ in a smooth bounded domain $G\subset E^n$. If the coefficients of (1) can be extended into a larger domain $G'\supset\bar{G}$ so that (i)–(iii) hold in $\bar{G}$ as well as G, then we say that (1) is nonsingular in G. Points of ∂G at which such an extension is not possible comprise the singular boundary S.

For $n=1$, these considerations will reduce to the well-known oscillation theory for the Sturm-Liouville equation

$$\frac{d}{dx}\left(a \frac{du}{dx}\right) + cu = 0 \tag{1'}$$

on an interval $G=(h, k)$. If $x=h$ is a singular point and $u(x)$ is a solution of (1′), then we have

DEFINITION 1. $u(x)$ is oscillatory at $x=h$ if, for every neighborhood $N(h)$, $u(x)$ has a zero in $G\cap N(h)$.

In extending this definition to solutions of (1), we restrict our attention to certain mild kinds of singularities on a single $n-1$ dimensional component S_i of S.

DEFINITION 2. We say that $u(x)$ is weakly oscillatory at S_i if, for every open set $H\supset S_i$, $u(x)$ has a zero in $H\cap G$.

DEFINITION 3. We say that $u(x)$ is strongly oscillatory at S_i if, for every $x\in S_i$ and every neighborhood $N(x)$, $u(x)$ has a zero in $G\cap N(x)$.

To simplify statements of theorems we shall assume that a change of variables has effected the canonical situation[1] in which

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[1] That this canonical form can generally be attained has been shown by Mihlin [3].

$$G \subset \{x \mid x_n > 0\},$$
$$S_i \subset \{x \mid x_n = 0\},$$
$$a_{in} = a_{ni} = 0 \quad \text{for} \quad i = 1, 2, \cdots, n-1.$$

A principal tool will be the following Sturmian theorem for elliptic equations [1; 2].

THEOREM 1. *Let u and v be solutions of the elliptic equations*

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial u}{\partial x_i}\right) + cu = 0, \tag{1}$$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\alpha_{ij} \frac{\partial v}{\partial x_i}\right) + \gamma v = 0 \tag{1a}$$

in a bounded domain $G \subset E^n$. If Γ is a domain satisfying $\bar{\Gamma} \subset G$ and if
(i) *the matrix $(\alpha_{ij} - a_{ij})$ is non-negative definite in $\bar{\Gamma}$,*
(ii) $c \geqq \gamma$,
(iii) $v = 0$ *on* $\partial\Gamma$,
then $u(x)$ must have a zero in $\bar{\Gamma}$.

In order to get oscillation theorems for (1), we shall let $\bar{x}$ denote the coordinates $(x_1, \cdots, x_{n-1})$ and make use of equations of the form

$$\frac{d}{dx_n}\left(\alpha(x_n) \frac{dv}{dx_n}\right) + \sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_j}\left(\alpha_{ij}(\bar{x}) \frac{\partial v}{\partial x_i}\right) + \gamma(x_n)v = 0 \tag{1b}$$

whose coefficients satisfy (i) and (ii) of Theorem 1. That is, we assume that in G

$$\text{(i}'\text{)} \qquad \sum_{i,j=1}^{n-1} (\alpha_{ij}(\bar{x}) - a_{ij}(x))\xi_i\xi_j \geqq 0 \quad \text{for all } (\xi_1, \cdots, \xi_{n-1}),$$
$$\alpha(x_n) - a_{nn}(x) \geqq 0;$$
$$\text{(ii}'\text{)} \qquad c(x) \geqq \gamma(x_n).$$

We shall also define $\mu_1(t)$ as the first eigenvalue of the boundary problem

$$-\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_j}\left(\alpha_{ij}(\bar{x}) \frac{\partial \phi}{\partial x_i}\right) = \mu\phi \quad \text{on} \quad G \cap \{x \mid x_n = t\}.$$
$$\phi = 0 \quad \text{on} \quad \partial G \cap \{x \mid x_n = t\},$$

and set $\mu_0 = \lim_{t\to 0} \sup \mu_1(t)$.

THEOREM 2. *If for some* $\epsilon > 0$ *the equation*

$$\frac{d}{dt}\left(\alpha(t)\frac{dw}{dt}\right) + [\gamma(t) - (\mu_0 + \epsilon)]w = 0 \tag{3}$$

is oscillatory at $t = 0$, *then every solution of* (1) *is weakly oscillatory at* S_i.

PROOF. From the definition of μ_0 there exists a subset Σ of S_i which satisfies $\bar{\Sigma} \subset S_i$ and for which the boundary problem

$$-\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial \theta}{\partial x_i}\right) = \nu\theta, \qquad \theta = 0 \text{ on } \partial\Sigma, \tag{4}$$

has smallest eigenvalue $\nu_1(t) \leqq \mu_0 + \epsilon$. Choose $\delta > 0$ so that the cylinder

$$\Gamma_\delta = \Sigma X\{x \mid 0 < x_n \leqq \delta\}$$

is contained in G. In Γ_δ we use separation of variables to solve the equation (1b) subject to the boundary conditions $v = 0$ on $\partial\Gamma \cap G$. One such solution is of the form $v_1(x) = \theta_1(\bar{x})w(x_n)$, where θ_1 is the eigenfunction of (4) corresponding to $\nu_1(x_n)$ and $w(t)$ is a solution of

$$\frac{d}{dt}\left(\alpha(t)\frac{dw}{dt}\right) - \nu_1(t)w + \gamma(t)w = 0,$$

$$w(\delta) = 0.$$

Since $-\nu_1(t) \geqq -(\mu_0 + \epsilon)$ and since (3) is oscillatory at $t = 0$, Sturm's comparison theorem assures us that $w(t)$ is also oscillatory at $t = 0$. Thus $v_1(x) = \theta_1(\bar{x})w(x_n)$ has a sequence of nodal domains of the form

$$\Gamma_k = \Sigma X\{x \mid \delta_k < x_n < \delta_{k-1}\}$$

where $\delta_k \downarrow 0$. By Theorem 1, $u(x)$ has a zero in each Γ_k. Thus $u(x)$ is weakly oscillatory at S_i.

THEOREM 3. *If for every real* M *the equation*

$$\frac{d}{dt}\left(\alpha(t)\frac{dw}{dt}\right) + [\gamma(t) + M]w = 0 \tag{5}$$

is oscillatory at $t = 0$, *then every solution of* (1) *is strongly oscillatory at* S_i.

PROOF. Let $x_0 = (\bar{x}_0, 0)$ be a point of S_i and suppose there exists a neighborhood of x_0 for which $N(x_0) \cap G$ contains no zeros of $u(x)$. Construct a cylinder $\Gamma' = \Sigma' X\{x \mid 0 < x \leqq \delta\}$ so that $\bar{x}_0 \in \Sigma'$ and $\Gamma' \subset N(x_0) \cap G$. Let λ_1 denote the smallest eigenvalue of

$$(4') \qquad -\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_j}\left(\alpha_{ij} \frac{\partial \psi}{\partial x_i}\right) = \lambda\psi, \qquad \psi = 0 \text{ on } \partial\Sigma'.$$

Using separation of variables to solve (1b) subject to $v=0$ on $\partial\Gamma' \cap G$, we again find a solution of the form $v(x) = \psi_1(\bar{x})w(x_n)$, where $\psi_1(\bar{x})$ is the eigenfunction of (4′) corresponding to λ_1 and $w(t)$ satisfies

$$(6) \qquad \frac{d}{dt}\left(\alpha(t) \frac{dw}{dt}\right) + [\gamma(t) - \lambda_1]w = 0.$$

By hypothesis, (6) is oscillatory at $t=0$, and $v(x)$ again defines a sequence of nodal domains of the form

$$\Gamma_k' = \Sigma' X\{x \mid \delta_k < x_n < \delta_{k-1}\}$$

where $\delta_k \downarrow 0$. By Theorem 1, $u(x)$ vanishes in every Γ_k'.

Bibliography

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