NONSOLVABLE IC GROUPS

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Introduction. In this paper we will consider the structure of a non-solvable group satisfying the condition which I will denote by IC.

DEFINITION. A finite group G is said to be an IC group if for any two nonincident subgroups H and K , $H \cap K$ has only cyclic Sylow groups.

REMARK. Two subgroups H and K are nonincident if $H \not\subseteq K$ and $K \not\subseteq H$.

It follows easily that both subgroups and factor groups of IC groups are themselves IC groups. The result we will prove is the following.

THEOREM. *If G is a nonsolvable IC group then G is isomorphic to one of the following:*

- (i) $SL(2, 4)$,
- (ii) $SL(2, 8)$,
- (iii) $SL(2, 5)$.

REMARK. In this context a p -group will be considered a Sylow group of itself. Thus for a p -group an IC group has $H \cap K$ cyclic for all nonincident subgroups H and K .

1. We want to consider the structure of a non-abelian IC 2-group. Let S be a 2-group which is an IC group. Suppose S is non-abelian. S contains at least two maximal subgroups M_1 and M_2 . Since S is IC, $M_1 \cap M_2$ is a cyclic group. We will show that $M_1 \cap M_2$ is the Frattini subgroup of S . Suppose this is not the case. Then $S/\Phi(S)$ has order ≥ 8 . $\Phi(S)$ being the intersection of maximal groups is cyclic and therefore contains a subgroup R of index 2 which is normal in S . S/R is not elementary abelian and is of order ≥ 16 . Consider any subgroup F/R of order 4 in S/R . Denote by “ $-$ ” subgroups of S modulo R . If $[\Phi(S)] \cap \bar{F} = \bar{1}$ then $[F\Phi(S)]$ is elementary abelian of order 8 when $\bar{F}$ is noncyclic. We can find an $\bar{M} \subseteq \bar{S}$ such that $(\bar{M} : [F\Phi(S)]) = 2$. Since $\bar{M}$ has more than one maximal group, we get a contradiction to $\bar{M}$ being an IC group by intersecting $[F\Phi(S)]$ with any other maximal subgroup of $\bar{M}$. If $\bar{F}$ is noncyclic and $[\Phi(S)] \subset \bar{F}$ we have that F is noncyclic containing $\Phi(S)$. Since $(F : \Phi(S)) = 2$ and $(S : \Phi(S)) \geq 8$ we get that F is contained in at least two maximal subgroups of S . This again contradicts S being IC. Therefore S/R has only cyclic 4-groups. By a standard result S/R is

Received by the editors April 26, 1963.

cyclic or quaternion. Since $(S/R: \{1\}) \geq 16$ and S/R is not cyclic we have a contradiction. Thus $(S/\Phi(S): \{1\}) = 4$ and $M_1 \cap M_2 = \Phi(S)$. Since every maximal subgroup of S must contain $\Phi(S)$ we have that S contains only three maximal subgroups. Denote them by M_1, M_2, M_3 . We put these conclusions in the form of a lemma.

LEMMA 1. *Let S be a non-abelian 2-group which is an IC group. Then S is a 2-generator group with cyclic Frattini subgroup.*

LEMMA 2. *If G is an IC group then G is p -normal for each $p \mid (G:1)$.*

PROOF. Let P be a p -Sylow group of G for any prime $p \mid (G:1)$. If P is abelian we are done. Thus assume P non-abelian. We must have that $Z(P)$ is cyclic. If not, then since P is an IC group we have that $P/Z(P)$ is cyclic. From this and a theorem of Zassenhaus we conclude that P would be abelian. This contradicts our assumption. Let T be another p -Sylow group of G with $T \neq P$. We will assume $Z(P) \subset T$ and will try to show that $Z(P) = Z(T)$. Since T is nilpotent we have $Z(P) \subset N_T(Z(P)) \subseteq T$.¹ Since every subgroup of a cyclic group is characteristic we have that either $N_T(Z(P)) = T$ or $N_T(Z(P))$ is noncyclic. If $N_T(Z(P)) = T$ we have that $Z(P) \triangleleft T$ and as is well known this yields $Z(P) = Z(T)$. If $N_T(Z(P))$ is noncyclic and not equal to T , then since $N_T(Z(P)) = T \cap N_G(Z(P))$ is noncyclic $T \subseteq N_G(Z(P))$ or $N_G(Z(P)) \subseteq T$. If the first holds then as before we conclude that $Z(P) = Z(T)$. The second alternative cannot hold since $P \subseteq N_G(Z(P))$ and $P \neq T$. Q.E.D.

LEMMA 3. *If G is an abelian IC group then either G is cyclic or a p group of type (p, p^n) or (p, p, p) .*

PROOF. This follows from the fundamental theorem of abelian groups and the property of being an IC group.

THEOREM 1. *Let G be an IC group. One of the following holds:*

- (1) *a 2-Sylow group of G is abelian,*
- (2) *a 2-Sylow group of G is quaternion,*
- (3) *G possesses a normal 2-complement.*

PROOF. The proof will be by induction on the order of G . Assume that the order G is even and the 2-Sylow group is not abelian or quaternion. As is well known then S must contain a noncyclic maximal subgroup. Denote it by M_1 . If $Z(S)$ is the center of S , we are done by 2-normality, Grün's theorem and induction when $N_G Z(S) \neq G$. There-

¹ $\subset$ means strict inclusion.

fore we may assume that $Z(S) \triangleleft G$. Consider $\bar{G} = G/Z(S)$. If $\bar{S}$ is quaternion of order 8 then all the maximal subgroups $\bar{M}_1, \bar{M}_2$ and $\bar{M}_3$ are cyclic. By a theorem of Zassenhaus we may assert that M_1, M_2 and M_3 are abelian groups. But then $\Phi(S)$ is the intersection of two abelian maximal groups and is thus in the center of S . This contradicts the fact that $S/Z(S)$ has order 8. If $S/Z(S)$ is not abelian we may apply induction to $\bar{G}$ and assert that $\bar{G}$ has a normal 2-complement $\bar{C}$. Since $Z(S)$ is a cyclic 2-Sylow subgroup of C we get a normal 2-complement in C which serves as a normal 2-complement in G . Thus we consider the case where $S/Z(S)$ is abelian. If $N_G(S) = S$, then by passing to $G/Z(S)$ and applying a theorem of Burnside we as above get a normal 2-complement in $G/Z(S)$ and hence in G . Thus we may assume $N_G(S) \supset S$.² Since M_1 is not cyclic and since G is an IC group it is not hard to see that $N_G(M_1) = S$. Since $N_G(S)$ is bigger than S we also have that $M_1 \cong M_2 \cong M_3$. By a similar argument $M_1/Z(S) \cong M_2/Z(S) \cong M_3/Z(S)$. Since all the maximal groups of abelian $S/Z(S)$ are isomorphic, we must have that $S/Z(S)$ is elementary abelian of order 4 or order 8. If $S/Z(S)$ is of order 8 we get a contradiction to Lemma 1. Thus $S/Z(S)$ is a Klein group. As before we conclude that M_1, M_2, M_3 are abelian. Since S is not quaternion and $Z(S)$ is cyclic there exist noncentral involutions. Therefore $M_i = Z(S) \times \epsilon_i$, where ϵ_i are noncentral involutions. There exists a subgroup R such that $(Z(S):R) = 2$ and R is normal in G . It is not hard to see that S/R is elementary abelian of order 8. This contradicts Lemma 1. Thus the theorem is proved. Q.E.D.

We would now like to study the nonsolvable IC groups. By Theorem 1 and the recent result of Feit and Thompson [2] as to the solvability of groups of odd order we see that the two Sylow groups are either abelian or quaternion. By Lemma 3 and a well-known result on cyclic 2-Sylow groups we may assert that the 2-Sylow group is either abelian of type $(2, 2^n)$ or elementary abelian of order 8 or quaternion. We will consider each case.

THEOREM 2.³ *If G is a nonsolvable IC group with an abelian 2-Sylow group then G is isomorphic to $SL(2, 2^2)$ or $SL(2, 2^3)$.*

PROOF. Consider $\bar{G} = G/G'$. Let G_0 be the inverse image of the 2-Sylow complement of $\bar{G}$ in G . Let S_0 be a 2-Sylow group of G_0 . Since G/G_0 is an abelian 2-group we must have that G_0 is a nonsolvable IC group. Suppose $G_0 \neq G$. Then there is a 2-Sylow group S of G such

² $\supset$ means strict inclusion.

³ The author is indebted to the referee for simplifying his proof of Theorem 2.

that $S_0 \subset S$ and $S \not\subseteq G_0$. Thus $S_0 \subseteq S \cap G_0$. This contradicts the nonsolvability of G_0 . We therefore have $G = G_0$.

Let S be a 2-Sylow group of G . Let $x \neq 1$ be any element of S . Since S is abelian we have that $S \subseteq C_G(X) \cap N_G(S)$. The IC property forces $C_G(x) \subseteq N_G(S)$ or $N_G(S) \subseteq C_G(x)$. If the latter case occurs, we must have that S intersection the center of $N_G(S)$ is not the identity. But by a theorem of Zassenhaus [4, p. 173], the transfer of G into S maps G onto this intersection. This contradicts the fact that $G = G_0$. Therefore for all $x \in S$, $x \neq 1$, we have $C_G(x) \subset N_G(S)$. Since a group of type $\{2, 2^n\}$ with $n > 1$ contains a unique noncyclic maximal subgroup and $N_G(S) \supset S$ we must have that S is elementary of order 4 or 8. Again by IC we must have that either $C_G(S) \subset C_G(x)$ or $C_G(x) \subseteq C_G(S)$. Suppose the first case occurs. If $|S| = 4$, since $(N_G(S) : C_G(S)) = 3$, we get that $C_G(x) = N_G(S)$. An application of Maschke's theorem on complete reducibility yields $N_G(S) = C_G(S)$. This contradicts the nonsolvability of G . If $|S| = 8$ we get that $C_G(x) = \text{SL}$, where $|L| \neq 1$. By Maschke's theorem we get that L normalizes a Klein group V . Thus $LV \cap S$ contradicts the IC property of G . Therefore we must have for all $x \neq 1$, $x \in S$, that $C_G(x) \subseteq C_G(S)$. But nonsolvable groups satisfying this property have been characterized by Feit in [3]. Such a group is $\text{SL}(2, 2^n)$ for some value of n . Since $|S|$ is 4 or 8 we get the result of the theorem. A close inspection verifies that $\text{SL}(2, 2^2)$ and $\text{SL}(2, 2^3)$ are indeed IC groups.

THEOREM 3. *If G is a nonsolvable IC group with a quaternion 2-Sylow group then $G \cong \text{SL}(2, 5)$.*

PROOF. By a theorem of Brauer [1], a group G with a quaternion 2-Sylow group contains a normal subgroup N of odd order such that G/N contains a unique involution. Let T/N be the unique and therefore normal subgroup of order 2 in G/N . Since T is solvable we must have that G/T is a nonsolvable IC group containing the Klein group as 2-Sylow group. Thus by Theorem 2 G/T must be isomorphic to $\text{SL}(2, 4)$. Since G is an IC group T has all cyclic Sylow groups. In particular T has a normal 2-complement R . If V is a noncyclic 4-group of G and S a 2-Sylow group containing V then $RV \cap S$ contradicts the IC property of G . Thus we must have that $|R| = 1$ and $|T| = 2$. By a result of Schur we get that either G is isomorphic to $\text{SL}(2, 5)$ or $A_5 \times T$. The second case is not possible by the structure of the 2-Sylow group. Thus G is isomorphic to $\text{SL}(2, 5)$. A check of the subgroup structure of this group verifies that $\text{SL}(2, 5)$ is an IC group.

Putting these results into one theorem we state:

THEOREM 4. *If G is a nonsolvable IC group then G is isomorphic to one of the following:*

- (i) $SL(2, 2^2)$,
- (ii) $SL(2, 2^3)$,
- (iii) $SL(2, 5)$.

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