GROUPS OF ORDER 1

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I. Introduction. Let a group be given by generators, x_i , of the free group F_n and defining relation, $R_i=1$, $i: 1, \dots, n$, and denoted by

$$G = (x_1, \dots, x_n; R_1, \dots, R_n) = (x; R).$$

One may wish to decide whether or not G is trivial ($G=1$). I shall discuss some properties of the trivial group and some decision techniques.

The symbols G, R, R_i will refer to those given above.

II. Simple words. In a free group F generated by the symbols $x_1, \dots, x_n$, of rank n , a word (element) $w_1(x_1, \dots, x_n) = w_1(x)$ is said to be simple (or a simple x -word) if there exist $n-1$ others, $w_2(x), \dots, w_n(x)$, which together with $w_1(x)$ generate F . If $v_1(x), \dots, v_n(x)$ is another generating set of x -words, the set $v_1(x), v_2(x), \dots, v_n(x)$ may not generate F . That is, a collection of n simple x -words may not be a generating set. I shall call the x -words $w_1, \dots, w_k, k \leq n$, of F a set of simple words, or briefly a simple set, if they can be completed to (a "complete set of simple words") $w_1, \dots, w_n$ that generate F .

The elements $R_1, \dots, R_n$ of F need not be simple or a simple set even if $G=1$. (Examples are given in §III, below.) Whether or not they are can be determined and if they are then $x_1, \dots, x_n$ are a simple set of R -words [6].

If they are not, the following rather obvious fact holds:

THEOREM 1. *Suppose the elements $R_1, \dots, R_n$ of the free group F of rank n , generated by $x_1, \dots, x_n$, have t conjugates, $R'_1, \dots, R'_t$, that can be written as a simple set (of g -words) in terms of the symbols $g_i = x_j^{k_i}, i: 1, \dots, t, j: 1, \dots, n$, and that those nonzero integers k_i belonging to one subscript j are relatively prime. Then the normal subgroup generated by $R_1, \dots, R_n$ in F is F itself.*

PROOF. Since the R'_i are a complete set of simple g -words, they generate $g_1, \dots, g_t$. As the latter, in their turn generate $x_1, \dots, x_n$ the conclusion follows.

Note that the expressions so gotten for the x_j in terms of the R'_i constitute a set of simple R' -words.

Presented to the Society, September 10, 1963; received by the editors June 21, 1963.

The converse is not trivial and is contained in

THEOREM 2. *Suppose $G = (x_1, \dots, x_n; R_1, \dots, R_n)$ is trivial, so that each x_i is the product $f_i(R_{ij}, i: 1, \dots, n, j: 1, \dots, s)$ of t conjugates $R_{ij} = z_{ij} R_i z_{ij}^{-1}$ of $R_1, \dots, R_n$ in F , the free group generated by the x_i . Then $f_1, \dots, f_n$ is a simple set of R_{ij} -words and possesses a completion by a set of R_{ij} -words, $f_{n+1}, \dots, f_t$ which vanish qua x -words (i.e., when the symbols R_{ij} are replaced by the x -words $R_{ij}(x)$).*

PROOF. If $(g_1, \dots, g_q; W_1, \dots)$ is a free product

$$G_1 * G_2 * \dots * G_k$$

of the groups G_i , then there is a set of simple g -words, $N(g_1), \dots, N(g_q)$ each of which is contained in one or another of the factors G_i . If W'_i is the expression of W_i written out in terms of the $N(g_j)$, then each W'_i belongs to some one of the factors (Grushko's theorem) [3].

Abbreviate $N(g_i)$ to s_i and suppose G above is isomorphic to the free group of rank n , F_n . Then one may take G_i as free cyclic, say $G_i = F_1 = F(y_i)$, with y_i a certain g -word, $i: 1, \dots, n$. It will follow that each s_i is some power $y_j^{k_i}$, $j: 1, \dots, n$, $i: 1, \dots, q$.

Take a fixed j . If s_i is y_j or y_j^{-1} , then this y_j is a simple g -word. Otherwise there must be several s_i that are powers of this same y_j , say

$$s_1 = y_1^{k_1}, s_2 = y_1^{k_2}, \dots, s_h = y_1^{k_h}$$

which together generate y_1 . Being powers of y_1 , these s_i commute modulo the relations of G . Since, on the one hand y_1 is a power product $p(s_1, \dots, s_h)$ of $s_1, \dots, s_h$ in G , and on the other hand $s_1, \dots, s_h$ are powers of y_1 in G , the nonzero integers among the k_i are relatively prime. Therefore, there exists an element C in the commutator subgroup F'_h of the free group F_h generated by the h symbols s_i such that $p(s_1, \dots, s_h)C$ is a simple s -word. In G , this word C is the identity, so one may put $y_1 = pC(s)$ (s denoting the collection of these s_i), whereby y_1 is expressed as a simple s -word. As any simple word in the free group F_h has a completion, so does $pC(s)$. Set

$$s_1^* = pC(s)$$

(this is y_1) and $s_2^*, \dots, s_h^*$ a set that completes it in F_h . Then, in G , $s_i^* = y_1^{r_i}$, $i: 1, \dots, h$, for some integers r_i . Furthermore, the words $s_i^* s_1^{*-r_i}$, $i: 2, \dots, h$ also complete s_1^* to a full set of free generators of F_h .

This procedure, carried out for each $j: 1, \dots, n$, yields n sets of

simple s -words. As no s_i appears in two of these sets, and as the total number of s -words gotten is q , they form a complete simple set of s -words in $F_q = F(s_1, \dots, s_q)$.

As $s_1, \dots, s_q$ are simple qua g -words, so is this new set. Note that it consists of the n words $y_1(g), \dots, y_n(g)$ that generate G (G is now isomorphic to F_n) and $q - n$ words, say $y_{n+1}(g), \dots, y_q(g)$, which are consequences of the defining relations of G .

Finally, to prove the theorem, view $g_1, \dots, g_q$ as symbols for those conjugates $R'_1, \dots, R'_q$ in $F = F(x_1, \dots, x_n)$ of $R_1, \dots, R_n$ of the theorems that appear in the expressions (assumed to exist) for $x_1, \dots, x_n$ in the normal subgroup that $R_1, \dots, R_n$ generate. Let $W_1, \dots$ be those g -words in the free group $F_q = F(g_1, \dots, g_q)$ which vanish identically when the g_i are replaced by the x -words $R'_i(x)$. Then the $y_j(g)$ are simple R' -words and so generate the same subgroup as $R'_1, \dots, R'_q$ do in $F_q = F(R'_1, \dots, R'_q)$.

Now the symbol R'_i stands for an x -word in the originally defined group G , so $y_j(R'(x))$ is an x -word there, say $z_j(x)$. The $z_j(x)$ generate a subgroup of $F = F(x_1, \dots, x_n)$. As the $R'_i(x)$ generate F , the complete set of simple R' -words $y_j(R'(x)) = z_j(x)$ also do. As $y_{n+1}(R'(x)), \dots, y_q(R'(x))$ vanish identically qua x -words, we have that $z_{n+1}(x) = z_{n+2}(x) = \dots = z_q(x) = 1$, identically in $F(x_1, \dots, x_n)$. Hence $z_1(x), \dots, z_n(x)$ generate this free group, with each $z_i(x)$ giving rise to a free factor.

The free factors $G_i = F(y_i)$, $i: 1, \dots, n$, posited above can be chosen in advance from among all possible factorizations of G into n free factors. So, if G is $F(x_1, \dots, x_n)$, the free cyclic factors $F(x_i)$ may be chosen as the G_i . Then $F(x_i) = F(z_i(x))$.

Because the only elements in $F(x_i)$ that generate it are x_i and its inverse, the element $z_i(x)$ must be one of these.

This concludes the proof of Theorem 2.

REMARK 1. Nielsen's method can also be used to prove this. The method is described in [1].

REMARK 2. One reason why Theorem 2 fails to give an effective procedure to decide whether G is trivial is exemplified in (b) of §III. The finite process that works in the first group mentioned under (a) of §III yields only a proper subgroup, H , of F , while the normal closure of H is, of course, F . This is verified easily by Nielsen's method referred to above. It follows that at least one conjugate $zR_i z^{-1}$, of some R_i , is needed to generate F in the sense of Theorem 2 that has longer x -length (the number of x -symbols in a word) than the x -length of R_i ; there is no general method to find this $zR_i z^{-1}$.

Let s_i again stand for a simple x -word in $F = F_n$, with

$$F = F(x_1, \dots, x_n) = F(s_1, \dots, s_n), \quad s_i = s_i(x) = s_i(x_1, \dots, x_n).$$

All the groups of §III have the property that R_i vanishes modulo s_i for some complete set $s_1, \dots, s_n$, of simple words. In all but the first example $s_1 = a, s_2 = b, s_3 = c$. Replacing R_i by s_i in groups with this property yields a factor-group. It is an interesting but difficult problem, posed by Magnus, to decide whether some R_i may always be replaced by an s_i whenever $G = 1$.

The statement that if $G = (x_1, \dots, x_n; R_1, \dots, R_n) = 1$ then $R_1, \dots, R_k$ can be replaced by $x_1, \dots, x_k$ for every $k: 1, \dots, n$ is obviously equivalent to either of the following two:

(1) Modulo $R_{k+1}, \dots, R_n$, the words $R_1, \dots, R_k$ are consequences of $x_1, \dots, x_k, k: 1, \dots, n$. In symbols: $R_i = V_i W_i$, with V_i vanishing modulo $x_1, \dots, x_k$, and W_i vanishing modulo $R_{k+1}, \dots, R_n; i: 1, \dots, k$.

(2) $R_{k+1}, \dots, R_n$ generate $x_{k+1}, \dots, x_n$ modulo $x_1, \dots, x_k$. In symbols: $x_i = V_i W_i$, with V_i vanishing modulo $R_{k+1}, \dots, R_n$, and W_i vanishing modulo $x_1, \dots, x_k; i: k+1, \dots, n$.

Of course, neither (1) nor (2) suffices to make G trivial. Are they necessary? I have no counterexample. However, when $n = 2$, these conditions are always met (up to an automorphism of $F = F_2$). This follows from two observations: in the free group F_2 , generated by x_1 and x_2 , any element of the commutator subgroup vanishes modulo either x_i ; if the normal subgroup R_1 and R_2 generate in F_2 is F_2 itself, then modulo commutators, $R_i = s_i$ for some pair of generators $s_i = s_i(x), i: 1, 2$. Thus one may write F_2 and the R_i in terms of the s_i , so that

$$G = (s_1, s_2; s_1 C_1, s_2 C_2)$$

with the C_i elements of the commutator subgroup of F_2 . As $s_i C_i$ vanishes modulo s_i , the conclusion follows.

This is a special case of restricting the integer k to the values $n - 1$ and n , with n arbitrary. Here too, the conditions (1) and (2) are always met (up to an automorphism of F_n). The reasons are entirely analogous to the case $n = 2$.

III. Examples. (a) If R_1 and R_2 are words in the free group F generated by a and b , form their cyclically reduced conjugates $T_1, \dots, T_k$. (For example, the cyclically reduced conjugates of the word abb are abb, bab, bba .) Add to this set all products $T_i T_j^{\pm 1}$ that are shorter than

one of the words T_i, T_j . Repeat the procedure until nothing new turns up.

For

$$R_1 = a^2b^3, \quad R_2 = a^3b^4$$

this yields a set of a, b -words containing the words a and b ; therefore

$$G = (a, b; a^2b^3, a^3b^4) = 1.$$

This result also takes care of the group

$$\begin{aligned} G^* &= (a, b; ab^{-2}aba^{-1}ba^{-1}b, ab^{-1}(ba^{-1}ba^{-1}b)^{-1}ab^{-2}a(ba^{-1}ba^{-1}b)^2) \\ &= (a, b; aC_1, bC_2), \end{aligned}$$

C_1, C_2 elements of the commutator subgroup of $F(a, b)$. When one applies automorphisms of the free group on a and b to the words aC_1 and bC_2 to reduce their a, b -length (a finite procedure [6]), one arrives at a^2b^3 and a^3b^4 . It follows that G^* is also trivial.

(b) The group

$$(a, b, c; bab^{-1}a^{-2}, cbc^{-1}b^{-2}, aca^{-1}c^{-2}) = (a, b, c; R_1, R_2, R_3)$$

is trivial [5]. The method just used for G^* fails to discover this.

This group is a special case of

$$(a, b, c; bab^{-1}a^{-t-1}, cbc^{-1}b^{-t-1}, aca^{-1}c^{-t-1})$$

with t a fixed integer. When t is greater than 1, this group is not trivial [4].

(c) Is the group

$$G = (a, b, c; a^{-1}b^2ab^{-3}, b^{-1}c^2bc^{-3}, c^{-1}a^2ca^{-3}) = (a, b, c; R_1, R_2, R_3)$$

trivial?

Observe that R_1 can be written as

$$u_1 v_1 u_1 v_1 u_1 v_1$$

with

$$\begin{aligned} u_1 &= (a^{-1}bab^{-1})^2b^{-1}, \\ v_1 &= b^2a^{-1}b^{-1}a. \end{aligned}$$

The corresponding statement for R_2 and R_3 gives three words $u_1(a, b)$, $u_2(b, c)$, $u_3(a, c)$; the R_i are consequences of these words. Then, factoring out the u_i from G gives

$$H = G/u_1, u_2, u_3 = (a, b, c; u_1, u_2, u_3).$$

This group H satisfies the conditions of Greendlinger's theorem [2], which means that every consequence of u_1, u_2, u_3 contains at least five symbols. It follows that G cannot be trivial.

One can investigate the more general group

$$G_k = (A, B, C; A^{-k}B^2A^kB^{-3}, B^{-k}C^2B^kC^{-3}, C^{-k}A^2C^kA^{-3})$$

along similar lines. For example, by noting that the mapping

$$B \rightarrow a^{-1}bab^{-1}$$

$$C \rightarrow b^{-1}cbc^{-1}$$

$$A \rightarrow c^{-1}aca^{-1}$$

is an isomorphism between G_2 and H , one finds that

$$G_2 = (a, b, c; a^{-2}b^2a^2b^{-3}, b^{-2}c^2b^2c^{-2}, c^{-2}a^2c^2a^{-3})$$

is not trivial.

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