ON THE GROUP GENERATED BY A FREE SEMIGROUP

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The following theorem answers a question proposed by E. Gräner¹ and A. H. Frey.

THEOREM. *A group generated by a two generator free semigroup need not be free.*

Let H be the free group on generators a and b , K the free semigroup on a and b and N the normal subgroup of H generated by $ab^{-1}ab^{-1}$. Let $G=H/N$. In particular, we will show that K is embedded in G by the natural embedding.

We will use c for a, b, a^{-1} , or b^{-1} . When c and d are used in the same word, (c, d) will represent $(a, b), (b, a), (a^{-1}, b^{-1})$ or (b^{-1}, a^{-1}) . If W is any word on a, b, a^{-1}, b^{-1} we say that W has n changes of sign if $W=P_1P_2^{-1} \cdots P_{n+1}^{(-1)^n}$ or $P_1^{-1}P_2 \cdots P_{n+1}^{(-1)^{n+1}}$, where each P_i is a nonempty word on a and b .

We remark that if the semigroup generated by a and b were not free in G then there would be two distinct words P_1 and P_2 on a and b representing the same member of G . Hence there would be a nontrivial word $P_1P_2^{-1}$ which would represent an element of N and have precisely one change of sign. The construction which follows implies that every representative of a nontrivial element of N has at least three changes of sign.

Let U be the collection of all sets B of words on a, a^{-1}, b, b^{-1} obeying the following rules:

1. B contains e (the empty word), $aa^{-1}, bb^{-1}, b^{-1}b, a^{-1}a$ (the inverse pairs) and $a^{-1}ba^{-1}b, ab^{-1}ab^{-1}, b^{-1}ab^{-1}a, ba^{-1}ba^{-1}$ (which we shall call the basic words).

2. If V_1 and V_2V_3 are members of B and V_1 and V_2 are nonempty, then $V_2V_1V_3 \in B$ (N.B. V_2 and V_3 need not be members of B).

Let $A = \bigcap U$ (i.e., the smallest set satisfying rules 1 and 2). Clearly $A \in U$ and every element of A can be constructed from the elements given in rule 1 by a finite number of applications of rule 2.

LEMMA 1. *A contains all products of formal conjugates of the basic words.*

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¹ Proposed in the University of Illinois problem book. Originally proposed in the doctoral dissertation of A. H. Frey.

PROOF. Rule 2 applied to inverse pairs V_2V_3 permits conjugation of V_1 . Rule 2 applied with V_3 empty permits multiplication.

COROLLARY. *Each element of N is represented by an element of A .*

DEFINITION. A word V on a, a^{-1}, b, b^{-1} is *reduced* unless V is of the form $Scc^{-1}T$.

Lemmas 2 through 4 will show that if W is an element of A and V is obtained from W by successive deletions of inverse pairs, V is an element of A and furthermore V is either empty or has at least three changes of sign. Trivially, we can consider the word $P_1P_2^{-1}$ of paragraph 3 to be reduced, hence not in A . But, in view of Lemma 1, the unique reduced word representing a member of N would be in A . This will prove the theorem.

We will prove Lemmas 2, 3 and 4 by strong induction on the length of the word. In each case the basis of the induction is trivial, and we will assume that the conclusion holds for words shorter than the word under consideration. We adopt the following convention: An equation of the form $V = V_2V_1V_3$ means $V \in A$, $V_2V_3 \in A$, $V_1 \in A$, V_1 and V_2 are nonempty.

LEMMA 2. *Every nonempty reduced word in A has at least three changes of sign.*

PROOF. Let V be a nonempty reduced word in A . If V is a basic word, V has 3 changes of sign. If V is reduced and not basic, then $V = V_2V_1V_3$. Therefore, V_1 is reduced and has, by the induction hypothesis, three changes of sign. Hence V does also.

LEMMA 3. *$cV \in A$ if and only if $Vc \in A$.*

PROOF. If $cV \in A$ and is an inverse pair or basic word, $Vc \in A$. If $cV = V_2V_1V_3 = cUV_1V_3$ then $cUV_3 \in A$ by our convention, $UV_3c \in A$ by inductive hypothesis on shorter words and $UV_1V_3c = Vc \in A$ by rule 2. The proof of the other implication is similar.

COROLLARY. *If $V_1V_2 \in A$ then $V_2V_1 \in A$.*

In proving the next lemma, we will use the following notation: $P * Q \rightarrow R$ means that $R \in A$ by an application of rule 2 to P and Q which are elements of A . $P - i \rightarrow R$ means that P is a word in A which is shorter than the word being considered in the inductive step and the inductive hypothesis may be applied to show that $R \in A$. $P - r \rightarrow R$ means that $P \in A$ and $P = V_2V_1V_3$ as in rule 2, with $R = V_2V_3$, whence $R \in A$.

LEMMA 4. *If $V = cc^{-1}S \in A$ then $S \in A$.*

PROOF. If V is an inverse pair, S is empty and in A . V cannot be a basic word. Hence we can assume $V = V_2 V_1 V_3$.

Case 1. $V_2 = cc^{-1}T$ so $V_2 V_1 V_3 = cc^{-1}T V_1 V_3 - r \rightarrow cc^{-1}T V_3 - i \rightarrow T V_3$ and $T V_3 * V_1 - 2 \rightarrow T V_1 V_3 = S$.

Case 2. $V_2 = c$, $V_2 V_1 V_3 = cc^{-1}R V_3$.

Subcase 1. V_1 and $V_2 V_3$ both arise from rule 1.

a. Both are inverse pairs so $V = cc^{-1}cc^{-1}$, $S = cc^{-1} \in A$.

b. V_1 is an inverse pair, $V_2 V_3$ a basic word, $V = cc^{-1}cd^{-1}cd^{-1}$, $S = cd^{-1}cd^{-1} \in A$.

c. $V_2 V_3$ is an inverse pair, V_1 is a basic word, $V = cc^{-1}dc^{-1}dc^{-1}$, $S = dc^{-1}dc^{-1} \in A$.

d. $V_2 V_3$ and V_1 are basic words, $V = cc^{-1}dc^{-1}dd^{-1}cd^{-1}$. $S = dc^{-1}dd^{-1}cd^{-1}$ arises from two applications of rule 2 to inverse pairs and hence is in A .

Subcase 2. $V_1 = U_2 U_1 U_3$, so $U_2 = c^{-1}T$ and $V_2 V_1 V_3 = cc^{-1}T U_1 U_3 V_3 - r \rightarrow c V_3$. But $U_2 U_1 U_3 = c^{-1}T U_1 U_3 - r \rightarrow c^{-1}T U_3$ and $c^{-1}T U_3 * c V_3 - 2 \rightarrow cc^{-1}T U_3 V_3 - i \rightarrow T U_3 V_3$. Hence $T U_3 V_3 * U_1 - 2 \rightarrow T U_1 U_3 V_3 = S$.

Subcase 3. $V_2 V_3 = U_2 U_1 U_3$, so $U_2 = cT$ and $U_2 U_1 U_3 = cT U_1 U_3 - r \rightarrow cT U_3$ and $cT U_3 * c^{-1}R - 2 \rightarrow cc^{-1}RT U_3 - i \rightarrow RT U_3$. Hence $RT U_3 * U_1 - 2 \rightarrow RT U_1 U_3 = S$.

COROLLARY. If $V_1 cc^{-1}V_2 \in A$, then $V_1 V_2 \in A$.

PROOF. Use corollary to Lemma 3.

This corollary asserts that reduction is permitted in A and the theorem is proved.

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