MEAN OSCILLATION OVER CUBES AND HÖLDER CONTINUITY

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A function $u=u(x)$, $x=(x_1, \cdots, x_n)$, is said to have bounded mean oscillation on a bounded cube C_0 if $u(x)$ is integrable over C_0 and there is a constant K such that for every parallel subcube C, and some constant a_C, the inequality

$$\int_C |u(x) - a_C| \, dx \leqq KR^n \tag{1}$$

holds, R being the edge length of C. Such functions have been investigated by John and Nirenberg [1]. Their result states that if $u(x)$ has bounded mean oscillation on C_0 and satisfies (1) then the function

$$\mu(\sigma) = \text{meas}\{ |u(x) - a_{C_0}| > \sigma \}$$

("meas" means Lebesgue measure) satisfies

$$\mu(\sigma) \leqq BR_0^n e^{-b\sigma/K},$$

where R_0 is the edge length of C_0 and B, b are constants depending only on n.

In this paper I show that if $u(x)$ satisfies an inequality of the form (1) with R^n replaced by $R^{n+\epsilon}$, $\epsilon>0$, then u is Hölder continuous with exponent ϵ (this condition is of course also necessary for Hölder continuity). Morrey's Lemma then follows as a simple corollary. The method of proof is essentially the same as that of John and Nirenberg and is based on the following decomposition lemma, a proof of which can be found in their paper.

LEMMA. *Let $u(x)$ be an integrable function on the bounded cube C_0 and let s be a positive number such that*

$$s \geqq R_0^{-n} \int_{C_0} |u(x)| \, dx. \tag{2}$$

There then exists a denumerable set of open disjoint parallel subcubes I_k $(k=1, 2, \cdots)$ such that

(i) $|u(x)| \leqq s$ *a.e. in* $C_0 - \sum_k I_k$,

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(ii) *the average*, u_k, *over* I_k *satisfies* $|u_k| \leqq 2^n s$,
(iii) $\sum_k R_k^n \leqq s^{-1} \int_{C_0} |u(x)|\, dx$ (R_k = *edge length of* I_k).

THEOREM. *Let* $u = u(x)$ *be an integrable function on a bounded cube* C_0. *Assume there exists a nondecreasing function* $K(R)$ *and a constant* ϵ, $0 < \epsilon \leqq 1$, *such that for every parallel subcube* C *and some constant* a_C *the inequality*

$$\int_C |u(x) - a_C|\, dx \leqq K(R) R^{n+\epsilon} \tag{3}$$

holds, R *being the edge length of* C. *Then there is a function* $v(x) = u(x)$ *a.e. in* C_0, *such that*

$$|v(x) - v(y)| \leqq K_1 K(|x - y|)\, |x - y|^\epsilon \tag{4}$$

holds for all points x, y *in* C_0, *with* K_1 *depending only on* ϵ *and* n.

The function $K(R)$ may tend to zero as $R \to 0$ in which case $v(x)$ is better than Hölder continuous.

PROOF. If inequality (3) holds then it will also hold with $K(R)$ replaced by the constant $K(R_0)$. We call this constant K.

Since

$$\left| \int_C u(x) dx - a_C R^n \right| \leqq \int_C |u(x) - a_C|\, dx,$$

it follows that u_C, the mean value of $u(x)$ over C, satisfies $|u_C - a_C| \leqq K R^\epsilon$. Hence

$$\int_C |u(x) - u_C|\, dx \leqq 2K R^{n+\epsilon}. \tag{5}$$

Let $\Gamma = \Gamma(K;\ \epsilon;\ R_0)$ be the class of all integrable functions $u(x)$ satisfying the condition (5) on some cube C_0 of edge length R_0. Let $\mu(\sigma) = \mu(\sigma;\ K;\ \epsilon;\ R_0)$ be defined by

$$\mu(\sigma) = \sup_{u \text{ in } \Gamma(K;\epsilon;R_0)} \operatorname{meas}\{\, |u(x) - u_{C_0}| > \sigma \}. \tag{6}$$

Now multiply both sides of (5) by an arbitrary positive constant K' and set $w(x) = K' u(x)$. It is clear that w satisfies

$$\int_C |w(x) - w_C|\, dx \leqq 2KK' R^{n+\epsilon}$$

and therefore $\mu(\sigma;\ K;\ \epsilon;\ R_0) = \mu(\sigma K';\ KK';\ \epsilon;\ R_0)$. Substituting σ/K' for σ in this equation, we get

$$\mu(\sigma; KK'; \epsilon; R_0) = \mu\left(\frac{\sigma}{K'}; K; \epsilon; R_0\right). \tag{7}$$

Next, perform a similarity transformation $y=(R'/R_0)x$ which carries the cube C_0 onto a cube C' of edge length R', and set $w(y) = u((R_0/R')y)$. $w(y)$ satisfies

$$\int_C |w(y) - w_C|\, dy \leqq 2K\left(\frac{R_0}{R'}\right)^{\epsilon} R^{n+\epsilon}$$

for every parallel subcube C of C' with edge length R. It easily follows that $\mu(\sigma; K(R_0/R')^{\epsilon}; \epsilon; R') = (R'/R_0)^n \mu(\sigma; K; \epsilon; R_0)$. Substituting $K(R'/R_0)^{\epsilon}$ for K gives

$$\mu(\sigma; K; \epsilon; R') = \left(\frac{R'}{R_0}\right)^n \mu\left(\sigma; K\left(\frac{R'}{R_0}\right)^{\epsilon}; \epsilon; R_0\right). \tag{8}$$

Let σ and s be arbitrary numbers such that

$$2^{-n}\sigma \geqq s \geqq R_0^{-n} \int_{C_0} |u(x)|\, dx.$$

From the decomposition lemma we then have

$$\begin{aligned} &\operatorname{meas}\{|u(x)| > \sigma; x \text{ in } C_0\} \\ &\qquad \leqq \sum_k \operatorname{meas}\{|u(x) - u_k| > \sigma - 2^n s; x \text{ in } I_k\}. \end{aligned} \tag{9}$$

If we assume, as we may, that $u_{C_0}=0$, (9) then implies

$$\mu(\sigma; K; \epsilon; R_0) \leqq \sum_k \mu(\sigma - 2^n s; K; \epsilon; R_k).$$

From (8) we then have

$$\mu(\sigma; K; \epsilon; R_0) \leqq \sum_k \left(\frac{R_k}{R_0}\right)^n \mu\left(\sigma - 2^n s; K\left(\frac{R_k}{R_0}\right)^{\epsilon}; K; \epsilon; R_0\right)$$

and from (7) we further deduce

$$\mu(\sigma; K; \epsilon; R_0) \leqq \sum_k \left(\frac{R_k}{R_0}\right)^n \mu\left((\sigma - 2^n s)\left(\frac{R_0}{R_k}\right)^{\epsilon}; K; \epsilon; R_0\right). \tag{10}$$

Statement (iii) in the decomposition lemma gives

$$\left(\frac{R_0}{R_k}\right)^{\epsilon} \geqq s^{\epsilon/n} M, \qquad M = R_0^{\epsilon}\left(\int_{C_0} |u(x)|\, dx\right)^{-\epsilon/n}. \tag{11}$$

Using the fact that μ is nonincreasing in σ, we then have from (10), (11) and (iii) of the decomposition lemma

$$\mu(\sigma) \leqq s^{-1}R_0^{-n}\int_{C_0} |u(x)|\,dx\cdot\mu((\sigma-2^n s)s^{\epsilon/n}M). \tag{12}$$

Set $\sigma=2^{n+1}s$. Then $(\sigma-2^n s)s^{\epsilon/n}M=s^{\epsilon/n}\cdot M/2\cdot\sigma$. Thus, if we set

$$s=\left(\frac{2}{M}\right)^{n/\epsilon}=2^{n/\epsilon}\cdot R_0^{-n}\int_{C_0}|u(x)|\,dx$$

we get

$$\mu(\sigma)\leqq 2^{-n/\epsilon}\mu(\sigma).$$

Therefore $\mu=0$ for $\sigma=2^{n/\epsilon+n+1}R_0^{-n}\int_{C_0}|u(x)|\,dx$, or in other words

$$|u(x)-u_{C_0}|\leqq 2^{n/\epsilon+n+2}K(R_0)R_0^{\epsilon}$$

a.e. in C_0. Therefore

$$|u(x)-u(y)|\leqq 2^{n/\epsilon+n+3}K(R_0)R_0^{\epsilon} \tag{13}$$

for almost all x and y in C_0. Since C_0 is an arbitrary cube and since any two points x, y with $|x-y|=R$ can be inclosed in a parallel subcube of edge length R the desired result follows from (13).

COROLLARY. *Let $u=u(x)$ have strong derivatives which are in L^p $(1\leqq p<\infty)$ on a bounded cube C_0. Assume there is a nondecreasing function $K=K(R)$ and a constant ϵ, $0<\epsilon\leqq 1$, such that for every parallel subcube C*

$$\int_C |\operatorname{grad} u(x)|^p\,dx\leqq K^p(R)R^{(n-p)+p\epsilon} \tag{14}$$

holds, R being the edge length of C. Then there is a function $v(x)=u(x)$ a.e. in C_0 such that

$$|v(x)-v(y)|\leqq K_2K(|x-y|)\,|x-y|^{\epsilon} \tag{15}$$

holds for all points x, y in C_0 and K_2 depends only on ϵ and n.

PROOF. It is a simple matter to prove the Wirtinger inequality

$$\int_C |u(x)-u_C|\,dx\leqq K_3R\int_C|\operatorname{grad} u(x)|\,dx, \tag{16}$$

with K_3 depending only on n. Applying the Hölder inequality to the right side of (16) we get

$$(17) \qquad \int_C |u(x) - u_C|\, dx \leqq K_3 R^{n+1-n/p} \left(\int_C |\operatorname{grad} u(x)|^p\, dx \right)^{1/p}$$

and the desired result follows from the previous theorem.

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Reference

1. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

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PHRAGMÉN-LINDELÖF THEOREMS FOR SECOND ORDER QUASI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Phragmén-Lindelöf theorems for uniformly elliptic partial differential equations have been the subject of several papers in recent years (see e.g. [**3**; **4**; **6**; **7**; **8**; **10**]). Here we are concerned with the Phragmén-Lindelöf theorem for second order quasi-linear elliptic equations of the form

$$(1) \qquad L[z] \equiv \sum a_{ij}(x, p) z_{x_i x_j} = f(x, z, p),$$

which *need not* be uniformly elliptic. The main result is Theorem **1** which roughly says that if $u(x)$ is a subfunction with respect to (1) in a domain D contained in a half space and if $u(x) \leqq 0$ on the boundary of D then either $u(x) \leqq 0$ throughout D or the maximum of $u(x)$ on a sphere of radius r is of order not less than r^η for some $\eta > 0$. Probably the most interesting feature of this theorem is that its proof essentially depends only on the behavior of the functions $a_{ij}(x, p)$ and $f(x, z, p)$ for $\sum p_i^2 \leqq 1$. For $f \equiv 0$ and dimension $n = 2$ it is shown that $\eta = 1$.

Let D be an unbounded domain contained in a half space of n-dimensional Euclidean space and let T be the domain in $2n$-dimen-

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